Mathematics G4402x
Modern Geometry

Assignment #5
Due November 11, 2013

1. Suppose G is a connected Lie group and H is any Lie group. If $\Phi, \Psi : G \to H$ are Lie group homomorphisms such that $\Phi_* = \Psi_* : \mathfrak{g} \to \mathfrak{h}$, show that $\Phi = \Psi$.

2. (a) Use Gram-Schmidt to show that every matrix in $SL(n, \mathbb{C})$ can be uniquely expressed as $A = BC$, where $B \in SU(n)$ and C is in the subgroup of $SL(n, \mathbb{C})$ consisting of upper-triangular matrices with positive real entries on the diagonal.

(b) Show that $SL(2, \mathbb{C})$ is diffeomorphic to $S^3 \times \mathbb{R}^3$ and hence is simply connected.

3. (a) Use Gram-Schmidt to show that every matrix in $SL(n, \mathbb{R})$ can be uniquely expressed as $A = BC$, where $B \in SO(n)$ and C is in the subgroup of $SL(n, \mathbb{R})$ consisting of upper-triangular matrices with positive entries on the diagonal.

(b) Show that $SL(2, \mathbb{R})$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

(c) Show that the universal cover of $SL(2, \mathbb{R})$ has infinite cyclic center.
(Careful! What is the center of $SL(2, \mathbb{R})$ itself?)

4. (a) Prove that any matrix in $SL(2, \mathbb{R})$ is either:
(i) *hyperbolic*, with eigenvalues $\lambda_1 \neq \lambda_2 \in \mathbb{R}$;
(ii) *parabolic*, with eigenvalues $\lambda_1 = \lambda_2 = \pm 1$; or
(iii) *elliptic*, with eigenvalues $\lambda_1 = \bar{\lambda}_2 \in U(1) \setminus \{\pm 1\}$.

(b) Show that $A \in SL(2, \mathbb{R})$ is in the image of the exponential map if and only if $\mathrm{tr}\, A > 0$, A is elliptic, or $A = -I$.

5. Let G be the universal covering group of $SL(2, \mathbb{R})$. Show that there is no *faithful representation* of G, that is, no injective homomorphism $\rho : G \to GL(n, \mathbb{R})$, as follows.

(a) Let $\rho_* : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ be the induced Lie algebra homomorphism, and show that $\phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ given by $\phi(A + iB) = \rho_*(A) + i\rho_*(B)$ is also a Lie algebra homomorphism.

(b) Show that there is a Lie group homomorphism $\Phi : SL(2, \mathbb{C}) \to GL(n, \mathbb{C})$ such that $\Phi_* = \phi$.

(c) Write down a diagram involving all five of the groups mentioned so far, show that it commutes, and use this to argue that ρ cannot be injective.

6. Let $N \subset M$ be an immersed submanifold, and let $X, Y \in VF(M)$ be such that for every $p \in N$, $X(p), Y(p) \in T_pN$. Show that then $[X, Y](p) \in T_pN$ as well.

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7. If $F : M \to N$ is a submersion, show that the connected components of the level sets of F form a foliation of M.

8. Prove that for any finite-dimensional vector space V, the wedge product is the *only* binary operation $\Lambda^p V^* \times \Lambda^q V^* \to \Lambda^{p+q} V^*$ satisfying (i) bilinearity; (ii) associativity; (iii) anti-commutativity, i.e. $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$; (iv) if p or $q = 0$, it is scalar multiplication; and (v) if $\alpha_1, \ldots, \alpha_k \in \Lambda^1 V^* = V^*$ and $v_1, \ldots, v_k \in V$, then

$$\alpha_1 \wedge \cdots \wedge \alpha_k(v_1, \ldots, v_k) = \det(\alpha_i(v_j)).$$

9. Show that for $X, Y \in VF(M)$, $\omega \in \Omega^k(M)$,

$$L_X \, i_Y \, \omega - i_Y \, L_X \, \omega = i_{[X,Y]} \, \omega.$$

10. Show that for $X \in VF(M)$, $\omega \in \Omega^k(M)$, $f \in C^\infty(M)$,

$$L_{fX} \, \omega = df \wedge i_X \, \omega + f L_X \, \omega.$$