Mathematics G4402x
Modern Geometry

Assignment #4

Due October 28, 2013

1. Let M be $\mathbb{R}^4$, regarded as the set of 2×2 matrices with real entries, and define an action of $\mathbb{R}$ on M by
$$t \cdot A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot A.$$
Find the infinitesimal generator.

2. Let $M = \mathbb{R}^n$ and consider vector fields $F : M \to \mathbb{R}^n$, $G : M \to \mathbb{R}^n$. Give an explicit formula for the Lie bracket $[F, G] : M \to \mathbb{R}^n$.

3. Show that there are exactly two 2-dimensional Lie algebras up to isomorphism. Describe each of them as a subalgebra of $\mathfrak{gl}(2, \mathbb{R})$.

4. Suppose M is a manifold embedded in $\mathbb{R}^N$. Show that there is a natural, bijective correspondence between vector fields $VF(M)$, as defined in class, and smooth maps $F : M \to \mathbb{R}^N$ such that for each $p \in M$, $F(p) \in T_pM \subset \mathbb{R}^N$.

5. Let $\gamma : (-a, b) \to M$ be an integral curve for the flow of $X \in VF(M)$. Recall that $\gamma'(t) := D_t\gamma(1)$.

(a) Show that either $\gamma'(t) = 0$ for all $t \in (-a, b)$ or $\gamma'(t) \neq 0$ for all $t \in (-a, b)$.

(b) Show that every nonconstant integral curve is an immersion.

6. If H is a Lie group and $F : G \to H$ a connected covering space (so that G is also a Lie group), show that the induced map $f : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism of Lie algebras.

7. Let G be a Lie group.

(a) Compute the derivative at (e, e) of the multiplication $m : G \times G \to G$.

(b) Compute the derivative at e of the inversion $i : G \to G$.

(c) Show that G is abelian if and only if i is a homomorphism.

(d) Show that if G is abelian, then its Lie algebra $\mathfrak{g}$ has zero bracket, $[X, X] = 0$.

(e) Give a counterexample to the converse statement.

8. The *support* of a vector field $X \in VF(M)$ is the closure of $\{p \in M \mid V(p) \neq 0\}$. If X has compact support, show that its flow is complete (that is, defined on all of $\mathbb{R} \times M$).

9. Show that the group of diffeomorphisms acts transitively on any connected manifold: that is, for any $p, q \in M$ connected, there exists a diffeomorphism $f : M \to M$ such that $f(p) = q$. Hint: first show that, if p, q lie in an embedded open ball, there is a compactly supported flow taking p to q.