Mathematics G4402x
Modern Geometry

Assignment #3

Due October 14, 2013

1. The *one-point compactification* $\overline{X}$ of a Hausdorff space X is the set $X \cup \{\infty\}$ with the topology whose open sets are the open subsets of X and those sets of the form $\{\infty\} \cup X \backslash C$ with $C \subset X$ compact. Show that a continuous map $f : X \to Y$ is proper if and only if it extends to a continuous map $\overline{f} : \overline{X} \to \overline{Y}$ with $\overline{f}(\infty) = \infty$.

2. If G is a Lie group acting smoothly (but not necessarily properly) on a manifold M, show that the stabilizer group $G_x = \{g \in G \,|\, gx = x\}$ of any $x \in M$ is a closed embedded Lie subgroup.

3. If $H \subset G$ is a closed embedded Lie subgroup which is normal in G, show that G/H is a Lie group and the projection $\pi : G \to G/H$ is a Lie group homomorphism. (The main issue is to show that multiplication and inversion are smooth.)

4. If $\phi : G_1 \to G_2$ is a smooth homomorphism of Lie groups (not necessarily proper), show that $\ker \phi$ is a closed embedded Lie subgroup. If ϕ is also surjective, show that there is an isomorphism of Lie groups $G_2 \cong G_1 / \ker \phi$.

5. Show that $\mathbf{CP}^1$ is diffeomorphic to S^2. Hint: express both as homogeneous spaces. Considerably harder for extra credit: show that $\mathbf{HP}^1$ is diffeomorphic to S^4.

6. Let $\mathrm{Fl}_k(\mathbf{C}^n)$ be the set of all *k-flags* in $\mathbf{C}^n$, that is, ordered k-tuples of mutually orthogonal 1-dimensional complex subspaces of $\mathbf{C}^n$. Show that $U(n)$ acts transitively on $\mathrm{Fl}_k(\mathbf{C}^n)$ and determine the stabilizer group of $(\langle e_1 \rangle, \dots \langle e_k \rangle)$. With what homogeneous space can $\mathrm{Fl}_k(\mathbf{C}^n)$ be identified? What is its dimension? Show that $\mathrm{Fl}_k(\mathbf{C}^n)$ admits a smooth surjection to $\mathrm{Gr}_k(\mathbf{C}^n)$. What are the fibers of this map?

7. Show that any injective Lie group homomorphism of compact connected Lie groups of the same dimension must be an isomorphism.

8. Show that there is an embedding $\mathbf{Z}_n \subset SU(n) \times U(1)$ such that

$$\frac{SU(n) \times U(1)}{\mathbf{Z}_n} \cong U(n).$$