# Mathematics G4402x
# Modern Geometry

## Assignment #2

Due September 30, 2013

**1.** (a) Show that a continuous injection $f : M \to N$ of topological spaces is a homeomorphism onto its image if and only if $U \subset M$ open implies $f(U) \subset f(N)$ open in the subspace topology.

(b) If $f : M \to N$ is an injective immersion of manifolds with $M$ compact, show that $f$ is an embedding.

**2.** (a) Show that any submersion is an open map, i.e. $U$ open implies $f(U)$ open.

(b) Show that any regular $k$-dimensional submanifold of a $k$-dimensional manifold must be an open subset.

**3.** If $N \subset M$ is a closed injectively immersed submanifold, not necessarily regular, show that the restriction map $C^\infty(M) \to C^\infty(N)$ on smooth functions need not be surjective. (We will later prove that it is surjective if $N$ is regular.)

**4.** If $\pi : M \to N$ is a smooth map of manifolds with $M$ compact, $N$ connected, and $D_x f$ an isomorphism for all $x \in M$, prove that $\pi$ is a covering map. Give a counterexample if $M$ is not compact.

**5.** Regard $\mathbb{RP}^2$ as the quotient space of $S^2$ obtained by identifying antipodal points. Define $F : \mathbb{R}^3 \to \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Since $F(v) = F(-v)$, this gives a well-defined map $\tilde{F} : \mathbb{RP}^2 \to \mathbb{R}^4$. Show that this is an embedding.

**6.** Let $S(n)$ be the set of symmetric $n \times n$ real matrices.

(a) Show that $S(n)$ is a real vector space of dimension $n(n+1)/2$, and that the map $f : GL(n, \mathbb{R}) \to S(n)$ given by $f(A) = A^T A$ is a submersion.

(b) Show that $O(n) = \{A \in GL(n, \mathbb{R}) \,|\, A^T A = I\}$ is a compact Lie group of dimension $n(n-1)/2$.

**7.** (a) Let $GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \,|\, \det A > 0\}$. Show that $GL^+(n, \mathbb{R})$ is path-connected, hence connected. Hint: consider the elementary row operations.

(b) Show that $SO(n) = O(n) \cap GL^+(n, \mathbb{R})$ is connected. Hint: apply Gram-Schmidt to a path.

**8.** If $S$ is any subset of a group $G$, let $S^2 = \{gh \,|\, g, h \in S\}$. If $U$ is a neighborhood of the identity $e$ in a Lie group $G$, show that there is another neighborhood $V$ of $e$ such that $V^2 \subset U$.