1. Let $E \to M$ be a complex vector bundle, $L \to M$ a complex line bundle.
   (a) If $\tilde{E} = E \otimes L$, show that there is a natural diffeomorphism $\mathbb{P}\tilde{E} \simeq \mathbb{P}E$, in terms of which the relevant tautological line bundles satisfy $\tilde{T} \simeq T \otimes \pi^* L$.
   (b) Deduce from (a) a formula for the Chern classes of $E \otimes L$ in terms of those of $E$ and $L$. What does it boil down to if the rank of $E$ is 2?

2. For $n > 1$, consider the embedding $I : \mathbb{CP}^1 \to \mathbb{CP}^n$ given by $[x, y] \mapsto [x^n, x^{n-1}y, \ldots, y^n]$.
   (a) If $h$ denotes the usual generator of $H^2(\mathbb{CP}^n)$ (i.e. the Poincaré dual of $\mathbb{CP}^n - 1$) show that $I^* h = nh$. Hint: wedge product is Poincaré dual to transverse intersection.
   (b) Show that $I^* T \mathbb{CP}^n$ and $N_{\mathbb{CP}^1/\mathbb{CP}^n}$ are nontrivial as complex vector bundles.

3. Let $V$ be a finite-dimensional complex vector space, $\mathbb{P}V$ the set of its one-dimensional complex subspaces. Let the incidence correspondence be
   \[ I = \{(v), [f] \in \mathbb{P}V \times \mathbb{P}V^* | f(v) = 0\} \]
   Show that $I$ is a smooth manifold and determine its Betti numbers. In fact, give generators and relations for $H^*(I)$ as an algebra over the real numbers.

4. If $E, \tilde{E} \to M$ are complex vector bundles of ranks $r, \tilde{r}$, use the Chern-Weil definition of Chern classes to show that $c_1(E \otimes \tilde{E}) = \tilde{r} c_1(E) + r c_1(\tilde{E})$.

5. (a) Show that the Grassmannian $\text{Gr}_k \mathbb{C}^n$ of $k$-dimensional subspaces of $\mathbb{C}^n$ is a complex manifold and express the complex vector bundle $T\text{Gr}_k \mathbb{C}^n$ in terms of the tautological rank $k$ vector bundle $E \to \text{Gr}_k \mathbb{C}^n$. Use this to show that $T\text{Gr}_k \mathbb{C}^n$ is nontrivial (e.g., by restricting to a projective space),
   (b) Show that every vector field on the Grassmannian must have a zero. Use this to give another proof that $T\text{Gr}_k \mathbb{C}^n$ is nontrivial.

6. If $E, \tilde{E} \to M$ are real oriented vector bundles, define an orientation on $E \oplus \tilde{E}$ and show that $e(E \oplus \tilde{E}) = e(E)e(\tilde{E})$. Also express $e(\mathbb{P}(E \oplus \tilde{E}))$ in terms of $e(E \oplus \tilde{E})$.

7. For a finite-dimensional complex vector space $V$, recall that $\Lambda^n V$ is the space of alternating multilinear maps $V^* \times \cdots \times V^* \to \mathbb{C}$. For a rank $n$ complex vector bundle $E \to M$, let $\Lambda^n E$ be the associated complex line bundle.
   (a) If $E = L_1 \oplus \cdots \oplus L_n$, show that $\Lambda^n E \cong L_1 \otimes \cdots \otimes L_n$.
   (b) Use the splitting principle to prove that $c_1(E) = c_1(\Lambda^n E)$.

CONTINUED OVERLEAF...
8. (a) Show that there is no immersion $\mathbb{CP}^3 \to \mathbb{R}^7$.
(b) Show that there is no immersion $\mathbb{CP}^5 \to \mathbb{R}^{13}$.

9. Let $E \to M$ be a real vector bundle with connection $\nabla$, and let $\nabla^k : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ be its extension to bundle-valued forms, as in the lecture.
(a) Show that for all $k$, $\nabla^{k+1} \circ \nabla^k = \wedge F_{\nabla}$.
(b) Show that $\Omega^0(M, E) \xrightarrow{\nabla^0} \Omega^1(M, E) \xrightarrow{\nabla^1} \Omega^2(M, E) \xrightarrow{\nabla^2} \Omega^3(M, E) \cdots$ is a complex if and only if $\nabla$ is flat, that is, has zero curvature.
(c) If so, define its cohomology to be the de Rham cohomology with local coefficients in $E$, denoted $H^k(M, E)$, and compute this for the flat connection on the Möbius strip (regarded as a line bundle over the circle) with holonomy $-1$. Hint: compute with differential forms on the universal cover.