Mathematics G4403y
Modern Geometry

Assignment #14
Due May 7, 2014

1. Let $E \to M$ be a complex vector bundle, $L \to M$ a complex line bundle.

(a) If $\tilde{E} = E \otimes L$, show that there is a natural diffeomorphism $\mathbb{P}\tilde{E} \simeq \mathbb{P}E$, in terms of which the relevant tautological line bundles satisfy $\tilde{T} \cong T \otimes \pi^* L$.

(b) Deduce from (a) a formula for the Chern classes of $E \otimes L$ in terms of those of E and L. What does it boil down to if the rank of E is 2?

2. For $n > 1$, consider the embedding $I : \mathbb{CP}^1 \to \mathbb{CP}^n$ given by $[x, y] \mapsto [x^n, x^{n-1}y, \ldots, y^n]$.

(a) If h denotes the usual generator of $H^2(\mathbb{CP}^n)$ (i.e. the Poincaré dual of $\mathbb{CP}^{n-1}$) show that $I^* h = nh$. Hint: wedge product is Poincaré dual to transverse intersection.

(b) Show that $I^* T\mathbb{CP}^n$ and $N_{\mathbb{CP}^1/\mathbb{CP}^n}$ are nontrivial as complex vector bundles.

3. Let V be a finite-dimensional complex vector space, $\mathbb{P}V$ the set of its one-dimensional complex subspaces. Let the *incidence correspondence* be

$$I = \{([v], [f]) \in \mathbb{P}V \times \mathbb{P}V^* \mid f(v) = 0\}.$$

Show that I is a smooth manifold and determine its Betti numbers. In fact, give generators and relations for $H^*(I)$ as an algebra over the real numbers.

4. If $E, \tilde{E} \to M$ are complex vector bundles of ranks $r, \tilde{r}$, use the Chern-Weil definition of Chern classes to show that $c_1(E \otimes \tilde{E}) = \tilde{r} c_1(E) + r c_1(\tilde{E})$.

5. (a) Show that the Grassmannian $\mathrm{Gr}_k \mathbb{C}^n$ of k-dimensional subspaces of $\mathbb{C}^n$ is a complex manifold and express the complex vector bundle $T\mathrm{Gr}_k \mathbb{C}^n$ in terms of the tautological rank k vector bundle $E \to \mathrm{Gr}_k \mathbb{C}^n$. Use this to show that $T\mathrm{Gr}_k \mathbb{C}^n$ is nontrivial (e.g., by restricting to a projective space),

(b) Show that every vector field on the Grassmannian must have a zero. Use this to give another proof that $T\mathrm{Gr}_k \mathbb{C}^n$ is nontrivial.

6. If $E, \tilde{E} \to M$ are real oriented vector bundles, define an orientation on $E \oplus \tilde{E}$ and show that $e(E \oplus \tilde{E}) = e(E) e(\tilde{E})$. Also express $e(\tilde{E} \oplus E)$ in terms of $e(E \oplus \tilde{E})$.

7. For a finite-dimensional complex vector space V, recall that $\Lambda^n V$ is the space of alternating multilinear maps $V^* \times \cdots \times V^* \to \mathbb{C}$. For a rank n complex vector bundle $E \to M$, let $\Lambda^n E$ be the associated complex line bundle.

(a) If $E = L_1 \oplus \cdots \oplus L_n$, show that $\Lambda^n E \cong L_1 \otimes \cdots \otimes L_n$.

(b) Use the splitting principle to prove that $c_1(E) = c_1(\Lambda^n E)$.

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8. (a) Show that there is no immersion $\mathbb{CP}^3 \to \mathbb{R}^7$.

(b) Show that there is no immersion $\mathbb{CP}^5 \to \mathbb{R}^{13}$.

9. Let $E \to M$ be a real vector bundle with connection ∇, and let $\nabla^k : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ be its extension to bundle-valued forms, as in the lecture.

(a) Show that for all k, $\nabla^{k+1} \circ \nabla^k = \wedge F_\nabla$.

(b) Show that $\Omega^0(M, E) \xrightarrow{\nabla^0} \Omega^1(M, E) \xrightarrow{\nabla^1} \Omega^2(M, E) \xrightarrow{\nabla^2} \Omega^3(M, E) \cdots$ is a complex if and only if ∇ is *flat*, that is, has zero curvature.

(c) If so, define its cohomology to be the *de Rham cohomology with local coefficients* in E, denoted $H^k(M, E)$, and compute this for the flat connection on the Möbius strip (regarded as a line bundle over the circle) with holonomy -1. Hint: compute with differential forms on the universal cover.