Mathematics G4403y
Modern Geometry

Assignment #13
Due April 23, 2014

1. (a) Homotopy invariance of degree: if M, N are compact and oriented of the same dimension, and if $f, g : M \to N$ are (smoothly) homotopic, show that $\deg f = \deg g$.

(b) A manifold M is said to be *parallelizable* if TM is trivial. In lecture, Poincaré-Hopf was used to show that S^n is not parallelizable for even $n > 0$. Reprove the same statement as follows. If S^n admits a vector field which is nowhere zero, show that the identity map on S^n is homotopic to the antipodal map $(x_0, \ldots, x_n) \mapsto (-x_0, \ldots, -x_n)$. For n even show that this map is also homotopic to the reflection $(x_0, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$. Then use (a).

(c) If n is odd, show that S^n admits a vector field which is nowhere zero.

(d) Show that any Lie group is parallelizable.

(e) Show that S^0, S^1, and S^3 are parallelizable.

Remark: more advanced methods show that S^n is parallelizable if and only if $n = 0$, 1, 3, or 7. See e.g. Milnor, *Annals of Math.* 68: 444.

2. (a) Let E, $\tilde{E}$ be rank n vector bundles on M, trivialized on the same open cover U_i, with transition functions f_{ij} and $\tilde{f}_{ij} : U_{ij} \to GL(n, \mathbb{R})$ respectively. Show that $\tilde{E} \cong E$ if and only if there exist smooth $g_i : U_i \to GL(n, \mathbb{R})$ so that on each U_{ij}, $\tilde{f}_{ij} = g_j f_{ij} g_i$.

(b) Define the *Möbius bundle* on $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ to be the rank 1 vector bundle trivialized on $U_1 = S^1 \setminus \{1\}$ and $U_2 = S^1 \setminus \{-1\}$ with transition function $f_{12}(z) = \mathrm{Im} z / |\mathrm{Im} z|$. Show that if S^1 is identified with $\mathbb{RP}^1$ in the usual way, then the Möbius bundle is isomorphic to the tautological line bundle.

3. (a) Show that the Möbius bundle is nontrivial.

(b) Show that the tautological line bundle on $\mathbb{RP}^n$ is nontrivial for all n.

4. Let $E \to M$ be a vector bundle, $s \in \Gamma(E)$ a nowhere vanishing section. Show that there is a connection ∇ on E with $\nabla s = 0$. Does this remain true if s has a zero? Why or why not?

5. If $E \to M$ is a vector bundle with connection ∇, $g : N \to M$ a smooth map, show that $F_{g^*\nabla} = g^* F_\nabla \in \Omega^2(N, g^*E)$.

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6. Regard $\mathbb{R}^n \times \mathbb{R}$ as a trivial rank 1 vector bundle over $\mathbb{R}^n$, and let a connection on this bundle be defined by $d + A$ for $A \in \Omega^1(\mathbb{R}^n)$. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a embedding with image C. Show that the parallel transport from $\gamma(a)$ to $\gamma(b)$ along C is multiplication by $\exp \int_C A$.

7. Let $E \to M$ be a rank n vector bundle on a connected manifold with connection ∇, $p \in M$ a basepoint. Choose an isomorphism $E_p \cong \mathbb{R}^n$ and define the *holonomy* of ∇ around a loop ℓ based at p to be the matrix representation of the parallel transport $E_x \to E_x$ along ℓ.

(a) Show that the set of holonomies around all loops based at p is a subgroup of $GL(n, \mathbb{R})$.

(b) If q is another basepoint, show that the holonomy subgroup there is conjugate to that at p.

8. (a) Show that every (real) vector bundle is isomorphic to its dual.

(b) If E is a complex vector bundle, let the *conjugate bundle* $\bar{E}$ be the same real manifold, but with every local trivialization ψ_i replaced by its composition with the map $\mathbb{C}^n \to \mathbb{C}^n$ given by $(z_1, \ldots, z_n) \mapsto (\bar{z}_1, \ldots, \bar{z}_n)$. Show that $\bar{E} \cong E^*$.

(c) Show, however, that a complex vector bundle need not be isomorphic to its dual.

9. (a) Show that the Euler class of a complex line bundle does not depend on the choice of a Hermitian structure.

(b) If L, L' are complex line bundles over M, show that $e(L \otimes L) = e(L) + e(L')$ and $e(L^*) = -e(L)$.