Mathematics G4403y
Modern Geometry

Assignment #12
Due April 9, 2014

1. Let M be a Riemannian 2-manifold. A contractible region in M bounded by geodesic segments is called a *geodesic polygon*.

(a) If M has nonpositive Gaussian curvature everywhere, prove that there are no geodesic polygons with less than 3 vertices. You may use the formula asserted in class:

$$\int_M G\,\mathrm{vol}_M + \int_{\partial M} \kappa(s)\,ds + \sum_{\text{angles}} (\pi - \theta_i) = 2\pi.$$

(b) Give examples of geodesic polygons with 0, 1, and 2 vertices on surfaces of non-negative curvature.

2. Let M be compact and oriented of dimension $2n-1$, and let $i : M \times S^1 \to \mathbb{R}^{2n+1}$ be an isometric immersion. Show that the image has at least one point of nonpositive Gaussian curvature. (Recall that the Gaussian curvature is the product of the principal curvatures.) Is the corresponding statement true for even-dimensional M?

3. (a) Show that a Riemannian manifold of constant sectional curvature is Einstein.

(b) Show that the converse is true for a connected manifold of dimension 3.

4. Let G be a Lie group with bi-invariant metric. Show that the Ricci curvature of G is given, for left-invariant $X, Y \in VF(G)$, by $R(X,Y) = -\frac{1}{4}\,\mathrm{tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y)$, where $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is defined as $\mathrm{ad}_X(Y) = [X,Y]$.

5. (a) For $p \in M$, show that the second-order Taylor series of g in normal coordinates centered at p is

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{kl} R_{iklj} x_k x_l + O(|x|^3).$$

Hint: Let $\gamma(t) = \exp_p tV$ and $J(t)$ a Jacobi field along γ, and compute the first few derivatives of $|J(t)|^2$ at $t = 0$ in two ways.

(b) For a surface, deduce from (a) a formula relating the Gaussian curvature at p to the areas of discs centered at p.

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6. (20 pts) *Riemannian symmetric spaces.*

(a) A *symmetric space* is a connected Riemannian manifold M having, for any $p \in M$, an isometric involution $\sigma : M \to M$ with p as an isolated fixed point. (Recall: *involution* means $\sigma \circ \sigma = \mathrm{id}$; *isolated* means it is the only fixed point in some neighborhood.) Show that $D_p\sigma = -\mathrm{id}$ and that such a σ is unique.

(b) Show that the model spaces $\mathbb{R}^n$, S^n, $\mathbb{H}^n$ are symmetric spaces.

(c) Show that on a symmetric space, $\nabla R = 0$.

(d) Show that a symmetric space is complete. Hint: use the involution to extend geodesics.

(e) Show that the group of isometries of M acts transitively on M.

(f) Assume that the group of isometries of M turns out to be a Lie group G acting smoothly on M. (This is always true but difficult to prove.) If K is the stabilizer of a point p, show that K is compact by identifying it with a closed subgroup of $O(n)$. Define an involution s of G by $s(g) = \sigma g \sigma$. Let $G^s = \{g \in G \mid s(g) = g\}$ and let G_0^s be its identity component. Show that $G_0^s \subset K \subset G^s$.

(g) Now let G be a Lie group, with left-invariant metric, and let $s : G \to G$ be a homomorphism with $s^2 = \mathrm{id}$. If K is compact and $G_0^s \subset K \subset G^s$, show that G/K has a Riemannian metric making it a symmetric space on which G acts by isometries.

(h) Show that $s : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ given by $s(A) = (A^T)^{-1}$ satisfies the requirements. What is K?

(i) Show that $s : U(n) \to U(n)$ given by conjugation by $\begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}$ satisfies the requirements. What is K? What is M?

(j) Show that any compact Lie group is a symmetric space.