Mathematics G4403y
Modern Geometry

Assignment #11

Due March 26, 2014

In what follows, M always denotes a Riemannian manifold with its Levi-Civita connection.

1. (a) In $\mathbb{R}^2$, show that the graph $y = f(x)$, oriented upward, has curvature

$$\kappa(x) = \frac{\ddot{f}}{(1+\dot{f}^2)^{3/2}}.$$

(Dots always denote derivatives with respect to t.)

(b) In $\mathbb{R}^3$, compute the second fundamental form of the graph $z = f(x, y)$ in terms of the isomorphism $T_pM \to \mathbb{R}^2$ given by projection on the horizontal.

(c) Extract formulas for the Gaussian and mean curvature of the graph M.

(d) Verify that the helicoid $z = x \tan y$ is a minimal surface.

2. (a) Show that any surface of revolution $M \subset \mathbb{R}^3$ can be parametrized by

$$(a(t)\cos\theta, a(t)\sin\theta, b(t))$$

with $\dot{a}^2 + \dot{b}^2 = 1$.

(b) Show that the Gaussian curvature of M is $-\ddot{a}/a$.

(c) Describe a surface of revolution in $\mathbb{R}^3$ that has constant Gaussian curvature 1 but does not have constant mean curvature.

3. Let $S \subset \mathbb{R}^n$ be an open set, $F : S \to \mathbb{R}$ a submersion, and $M = F^{-1}(0)$. Show that the second fundamental form is given by

$$II(V, W) = -\sum_{i,j} \frac{V^i W^j \partial_i \partial_j F}{|\nabla F|^2} \nabla F,$$

where $V = \sum_i V^i \partial_i$, $W = \sum_j W^j \partial_j$, and ∇F is the classical gradient.

4. (a) Show that if two (4,0)-tensors have the symmetries of the Riemann curvature tensor and agree on expressions of the form (X, Y, Y, X), then they are equal.

(b) Suppose that M has constant sectional curvature K. Show that the Riemann curvature tensor is given in coordinates by

$$R = K \sum_{i,j,k,l} (g_{il}g_{jk} - g_{ik}g_{jl})\, dx_i \otimes dx_j \otimes dx_k \otimes dx_l.$$

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5. (a) Let G be a Lie group with a bi-invariant metric, X, Y, Z left-invariant vector fields. Show that $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$.

(b) Show that any Lie subgroup of G is totally geodesic.

(c) Show that the sectional curvatures of G are all nonnegative: indeed, if X and Y are an orthonormal basis for U, show that $K(U) = \frac{1}{4}|[X,Y]|^2$.

(d) If G is connected, show that it is flat if and only if it is abelian.

6. Show that the usual diffeomorphism $SU(2) \to S^3$ takes the bi-invariant metric to the round metric.

7. (a) If M is connected, show that an isometry $f : M \to N$ is determined by its value and derivative at any point. That is, if another isometry $\tilde{f}$ satisfies $\tilde{f}(p) = f(p)$ and $D_p\tilde{f} = D_p f$, then $\tilde{f} = f$.

(b) Let M be a connected n-dimensional manifold on which a Lie group G acts by isometries, with no element besides e acting as the identity. Show that $\dim G \leq n(n+1)/2$, with equality only if M is of constant sectional curvature. Give examples where equality holds.