# Mathematics G4403y
# Modern Geometry

**Assignment #10**
Due March 5, 2014

**0.** (Optional for extra credit)

(a) Let $M$ be a smooth manifold, $X \in VF(M)$. Show that there exists a positive smooth $f : M \to \mathbb{R}$ so that the flow of $fX$ is globally defined.

(b) An open $U \subset \mathbb{R}^n$ is *star-shaped* if there exists $\mathbf{p} \in U$ so that for all $\mathbf{q} \in U$, the line segment from $\mathbf{p}$ to $\mathbf{q}$ is contained in $U$. Prove that any star-shaped set is diffeomorphic to $\mathbb{R}^n$. Hint: if $X(\mathbf{q}) = \mathbf{q} - \mathbf{p}$, for a suitably chosen $f$, construct a diffeomorphism taking the flow of $fX$ to the flow of $X$.

In the remaining problems, $M$ always denotes a Riemannian manifold with its Levi-Civita connection.

**1.** A subset $S$ of $M$ is *(geodesically) convex* if for each $p, q \in S$, there is a minimizing geodesic, unique in $M$, lying wholly in $S$. Show that every point $p$ has an open convex neighborhood, as follows.

(a) Let $V$ be a totally normal neighborhood. For $\varepsilon$ small enough that $\exp_p B_{3\varepsilon}(0) \subset V$, define $W_\varepsilon = \{(q, v) \in TM \mid d(p, q) < \varepsilon, v \in T_qM, |v| = 1\}$ and let $f : W_\varepsilon \times (-2\varepsilon, 2\varepsilon) \to \mathbb{R}$ be given by

$$f(q, v, t) = d(\exp_q tv, p)^2.$$

Show that $f$ is smooth. Hint: although $d$ is not smooth in general (even on a circle), you can calculate in normal coordinates centered at $p$.

(b) Show that for small enough $\varepsilon$, $\partial^2 f/\partial t^2 > 0$ on $W_\varepsilon \times (-2\varepsilon, 2\varepsilon)$. Hint: calculate $f(p, v, t)$ explicitly.

(c) If $q_1, q_2 \in \exp_p B_\varepsilon(0)$ and $\gamma$ is a minimizing geodesic from $q_1$ to $q_2$, show that $d(\gamma(t), p)$ attains its maximum at one of the endpoints.

(d) Show that $\exp_p B_\varepsilon(0)$ is convex.

**2.** (a) Show that the intersection of convex sets (in the sense above) is convex.

(b) Use problem **0** to show that an open convex set is diffeomorphic to $\mathbb{R}^n$.

(c) Prove the result promised last semester: every smooth manifold has a good cover.

**CONTINUED OVERLEAF...**

**3.** (a) Let $\Gamma : [0,1] \times [0,1] \to M$ be a smooth map, and let $\gamma_u(v) = \Gamma(u,v)$. If $\gamma_0$ is a geodesic, $\Gamma(u,0) = p$, and $\Gamma(u,1) = q$, show that the "first variation" $\frac{d}{du}\ell(\gamma_u)|_{u=0} = 0$.

(b) Give an example to show that this is false if $\Gamma(u,0)$ or $\Gamma(u,1)$ are not constant.

(c) Now let $\Gamma : [0,1] \times S^1 \to M$ be a smooth map, $\gamma_u(v) = \Gamma(u,v)$, such that $\gamma_0$ is a (closed) geodesic. Show that $\frac{d}{du}\ell(\gamma_u)|_{u=0} = 0$. Notice that there are no constraints on any $\Gamma(u, v_0)$.

**4.** Let $M \subset \mathbb{R}^3$ be a surface of revolution, parametrized by $(t,\theta) \mapsto (a(t)\cos\theta, a(t)\sin\theta, b(t))$.

(a) Compute the metric elements in $(t,\theta)$ coordinates.

(b) Compute the Christoffel symbols in $(t,\theta)$ coordinates.

(c) Show that each *meridian* $\theta = \theta_0$ is a geodesic.

(d) Give necessary and sufficient conditions for a *latitude* $t = t_0$ to be a geodesic.

**5.** A *divergent curve* in $M$ is a smooth map $\gamma : [0,\infty) \to M$ which eventually leaves any compact set $C \subset M$, that is, there exists $t_C$ so that $t > t_C$ implies $\gamma(t) \notin C$. Show that $M$ is complete if and only if all divergent curves have infinite arclength.

**6.** Show that a homogeneous space $M = G/H$ on which $G$ acts by isometries is complete.

**7.** Let $G$ be a Lie group equipped with a bi-invariant metric, $\mathfrak{g}$ its Lie algebra of left-invariant vector fields.

(a) For any $X, Y, Z \in \mathfrak{g}$, show that

$$\langle [X,Y], Z\rangle = -\langle Y, [X,Z]\rangle.$$

Hint: let $\gamma(t)$ be the flow of $X$ from the identity, let $\mathrm{Ad}_g : G \to G$ be $\mathrm{Ad}_g(h) = ghg^{-1}$, and compute $\frac{d}{dt}\langle \mathrm{Ad}_{\gamma(t)}Y, \mathrm{Ad}_{\gamma(t)}Z\rangle$.

(b) Show that $\nabla_X Y = \frac{1}{2}[X,Y]$.

(c) Show that the geodesics through the identity are precisely the one-parameter subgroups, and that arbitrary geodesics are (right or left) translates of one-parameter subgroups.

(d) Show that $\exp = \exp$, i.e. the Lie and Riemannian exponential maps coincide.

(e) Show that $SL(2,\mathbb{R})$ cannot admit a bi-invariant metric.