Mathematics G4402x
Modern Geometry

Assignment #1
Due September 16, 2013

1. Show that a topological manifold is path-connected if and only if it is connected.

2. (Real projective space)

(a) Show that there exists a homeomorphism $f : \mathbb{R}^{n+1} \setminus \{0\} \to S^n \times \mathbb{R}$ so that the diagram below commutes:

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xrightarrow{f} & S^n \times \mathbb{R} \\ \pi \downarrow & & \downarrow p \\ \mathbb{RP}^n & \xleftarrow{\pi|_{S^n}} & S^n \end{array}$$

Here π is the quotient map and p is projection on the first factor.

(b) Characterize the inverse image in $S^n \times \mathbb{R}$ of an open set in $\mathbb{RP}^n$.

(c) Use this to show that $\mathbb{RP}^n$ is Hausdorff, second countable, and compact.

3. Show that compositions of smooth maps of manifolds are smooth.

4. (Bump functions)

(a) The following function $f : \mathbb{R} \to \mathbb{R}$ is smooth:

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(b) There exists a smooth $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) > 0$ if $x \in (-1, 1)$ but $g(x) = 0$ otherwise.

(c) There exists a smooth $h : \mathbb{R} \to [0, 1]$ such that $h(x) = 0$ if $x \leq 0$ but $h(x) = 1$ if $x \geq 1$. Hint: integrate.

(d) For any $\varepsilon > 0$, there exists a smooth $\psi : \mathbb{R}^n \to [0, 1]$ such that $\psi(x) = 0$ if $|x| \geq 2\varepsilon$ but $\psi(x) = 1$ if $|x| \leq \varepsilon$.

(e) If M is a smooth manifold, $p \in U$ open in M, there exists a smooth $\phi : M \to \mathbb{R}$ such that $\phi(x) = 0$ if $x \notin U$ but $\phi(x) = 1$ on some neighborhood V of p.

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5. (Stereographic projection) Let $U_- = S^2 \setminus \{(0,0,1)\}$, and define $\phi_- : U_- \to \mathbb{R}^2$ by $\phi_-(x,y,z) = (\bar{x}, \bar{y})$, where $(\bar{x}, \bar{y}, 0)$ is the unique point with third coordinate zero on the line through (x,y,z) and $(0,0,1)$. Let $U_+ = S^2 \setminus \{(0,0,-1)\}$ and define ϕ_+ similarly. (See the diagrams on p. 59 of Boothby or p. 29 of Lee.) Give explicit formulas for ϕ_- and ϕ_+ and show that they are homeomorphisms defining compatible charts and hence a smooth structure on S^2. (Extra credit: generalize to S^n.)

6. For $M \xrightarrow{\phi} N \xrightarrow{\psi} P$ smooth maps of manifolds, show that push-forward is covariant: $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

7. Show that if a nonempty m-manifold is diffeomorphic to an n-manifold, then $m = n$.