1. Let \( F = (F_1, \ldots, F_n) \). Since \( M \) is nonempty and compact, \( F_1 \in C^\infty(M) \) takes a maximum, say at \( p \in M \). Let \( \phi : U \to V \) be a chart with \( \phi(p) = 0 \) and let \( G = F \circ \phi^{-1} \); then \( G_1 = F_1 \circ \phi^{-1} : V \to \mathbf{R} \) has \( \partial G_1 / \partial x_i = 0 \) at \( p \) for all \( i \), so \( D_0 G \) is singular, hence so is \( D_p F \) by the chain rule.

2. If \( Y \) belongs to \( T_p N \) and \( f \) vanishes on \( N \), then \( Y f = Y(f|_N) = Y 0 = 0 \). Conversely, if \( Y f = 0 \) for every \( f \) vanishing on \( N \), choose a slice chart \( \phi : U \to V \) so that \( \phi(p) = 0 \) and \( \phi(N) = (\mathbf{R}^n \times 0) \cap V \). Let \( \psi \) be a bump function compactly supported on \( V \) with \( \psi = 1 \) near \( 0 \), and let \( x_i : V \to \mathbf{R} \) be the \( i \)-th coordinate function. Then \( (x_i \psi) \circ \phi \) extends to a smooth function \( f_i : M \to \mathbf{R} \) vanishing on \( N \) when \( i > n \), so if \( \phi_* Y = \sum a_i \partial / \partial x_i \), we have \( 0 = Y f_i = (\phi_* Y)(x_i \psi) = a_i \) when \( i > n \), that is, \( \phi_* Y \in T_0(\mathbf{R}^n \times 0) \) and hence \( Y \in T_p N \).


4. If \( k \) is odd, then \( \nu^2 = (-1)^k \nu^2 = 0 \), so \( \nu^n = 0 \) and \( L_X(\nu^n) = 0 \). But if \( k \) is even, since \( L_X \) is a derivation, \( L_X(\nu \wedge \nu) = L_X \nu \wedge \nu + \nu \wedge L_X \nu = ((-1)^k + 1)\nu \wedge L_X \nu = 2\nu \wedge L_X \nu \). Assume it for \( n \) by induction; then \( L_X(\nu^{n+1}) = L_X(\nu^n \wedge \nu) = n\nu^{n-1} \wedge L_X \nu \wedge \nu + \nu^n \wedge L_X \nu = ((-1)^k n + 1) \nu^n \wedge L_X \nu = (n+1) \nu^n \wedge L_X \nu \).

5. We may assume \( u \in \mathbf{R}^3 \setminus 0 \). There, both \( X \) and \( Y \) are along the field of planes tangent to the spheres centered at \( 0 \); this plane field is integrable, hence involutive, so \( Z \) is also along it.

6. 

\[
\chi(M \times N) = \sum_k (-1)^k \dim H^k(M \times N)
\]

\[
= \sum_k (-1)^k \left( \bigoplus_{i+j=k} H^i(M) \otimes H^j(N) \right)
\]

\[
= \sum_k (-1)^k \sum_{i+j=k} \dim H^i(M) \dim H^j(N)
\]

\[
= \sum_k \sum_{i+j=k} (-1)^{i+j} \dim H^i(M) \dim H^j(N)
\]

\[
= \left( \sum_i (-1)^i \dim H^i(M) \right) \left( \sum_j (-1)^j \dim H^j(N) \right)
\]

\[
= \chi(M) \chi(N).
\]
7. Let \( v_1, \ldots, v_n \) be a basis for \( T_eG \), \( v^1, \ldots, v^n \) the dual basis, let \( \phi_g : G \to G \) be left multiplication by \( g \), and let \( \omega \in \Omega^n(G) \) be given by \( \omega_g := \phi_g^* v^1 \wedge \cdots \wedge v^n \). This is a smooth form, as on the corresponding left-invariant vector fields \( X_i \) we have \( \omega(X_1, \ldots, X_n) = 1 \), the constant function, hence \( \omega(V_1, \ldots, V_n) \) is smooth for arbitrary \( V_i \in VF(G) \), which must be \( C^\infty \) linear combinations of \( X_1, \ldots, X_n \). But it is also nowhere vanishing, so \( G \) is orientable.

8. Since \( \dim g = \dim G > 0 \), there exists a nonzero left-invariant vector field, which is nowhere vanishing, and also \( G \) is orientable by the previous problem, so \( \chi(G) = 0 \) by the Poincaré-Hopf Index Theorem.