Mathematics G4402x
Modern Geometry

Answers to Final Exam

December 16, 2013

1. Let $F = (F_1, \ldots, F_n)$. Since M is nonempty and compact, $F_1 \in C^\infty(M)$ takes a maximum, say at $p \in M$. Let $\phi : U \to V$ be a chart with $\phi(p) = 0$ and let $G = F \circ \phi^{-1}$; then $G_1 = F_1 \circ \phi^{-1} : V \to \mathbf{R}$ has $\partial G_1/\partial x_i = 0$ at 0 for all i, so $D_0 G$ is singular, hence so is $D_p F$ by the chain rule.

2. If Y belongs to $T_p N$ and f vanishes on N, then $Yf = Y(f|_N) = Y0 = 0$. Conversely, if $Yf = 0$ for every f vanishing on N, choose a slice chart $\phi : U \to V$ so that $\phi(p) = 0$ and $\phi(N) = (\mathbf{R}^n \times 0) \cap V$. Let ψ be a bump function compactly supported on V with $\psi = 1$ near 0, and let $x_i : V \to \mathbf{R}$ be the ith coordinate function. Then $(x_i\psi) \circ \phi$ extends to a smooth function $f_i : M \to \mathbf{R}$ vanishing on N when $i > n$, so if $\phi_* Y = \sum a_i\, \partial/\partial x_i$, we have $0 = Yf_i = (\phi_* Y)(x_i \psi) = a_i$ when $i > n$, that is, $\phi_* Y \in T_0(\mathbf{R}^n \times 0)$ and hence $Y \in T_p N$.

3. $L_X L_Y Z - L_Y L_X Z = [X, [Y, Z]] - [Y, [X, Z]] = [X, [Y, Z]] + [Y, [Z, X]] = -[Z, [X, Y]] = [[X, Y], Z] = L_{[X,Y]} Z$, where the third equality is the Jacobi identity.

4. If k is odd, then $\nu^2 = (-1)^{k^2} \nu^2 = 0$, so $\nu^n = 0$ and $L_X(\nu^n) = 0$. But if k is even, since L_X is a derivation, $L_X(\nu \wedge \nu) = L_X \nu \wedge \nu + \nu \wedge L_X \nu = ((-1)^{k^2} + 1) \nu \wedge L_X \nu = 2\nu \wedge L_X \nu$. Assume it for n by induction; then $L_X(\nu^{n+1}) = L_X(\nu^n \wedge \nu) = n\nu^{n-1} \wedge L_X \nu \wedge \nu + \nu^n \wedge L_X \nu = ((-1)^{k^2} n + 1)\, \nu^n \wedge L_X \nu = (n+1)\, \nu^n \wedge L_X \nu$.

5. We may assume $\mathbf{u} \in \mathbf{R}^3 \setminus 0$. There, both X and Y are along the field of planes tangent to the spheres centered at 0; this plane field is integrable, hence involutive, so Z is also along it.

6.

$$\begin{aligned}
\chi(M \times N) &= \sum_k (-1)^k \dim H^k(M \times N) \\
&= \sum_k (-1)^k \dim \bigoplus_{i+j=k} H^i(M) \otimes H^j(N) \\
&= \sum_k (-1)^k \sum_{i+j=k} \dim H^i(M) \dim H^j(N) \\
&= \sum_k \sum_{i+j=k} (-1)^{i+j} \dim H^i(M) \dim H^j(N) \\
&= \left(\sum_i (-1)^i \dim H^i(M) \right) \left(\sum_j (-1)^j \dim H^j(N) \right) \\
&= \chi(M)\chi(N).
\end{aligned}$$

7. Let $v_1, \ldots, v_n$ be a basis for T_eG, $v^1, \ldots, v^n$ the dual basis, let $\phi_g : G \to G$ be left multiplication by g, and let $\omega \in \Omega^n(G)$ be given by $\omega_g := \phi_{g^{-1}}^* v^1 \wedge \cdots \wedge v^n$. This is a smooth form, as on the corresponding left-invariant vector fields X_i we have $\omega(X_1, \ldots, X_n) = 1$, the constant function, hence $\omega(V_1, \ldots, V_n)$ is smooth for arbitrary $V_i \in VF(G)$, which must be C^∞ linear combinations of $X_1, \ldots, X_n$. But it is also nowhere vanishing, so G is orientable.

8. Since $\dim \mathfrak{g} = \dim G > 0$, there exists a nonzero left-invariant vector field, which is nowhere vanishing, and also G is orientable by the previous problem, so $\chi(G) = 0$ by the Poincaré-Hopf Index Theorem.