Mathematics G4402x
Modern Geometry

Final Examination
December 16, 2013

Attempt all eight problems. On each page of your blue book, write the number of the problem you work on inside a *circle*. You may use any results from lectures or homework, but be sure to refer to them clearly and state the source. In grading the exam, I will emphasize accuracy, brevity, and clarity. Good luck!

1. Let M be a nonempty compact manifold of dimension $n > 0$. Show that there is no submersion $F : M \to \mathbf{R}^n$.

2. Let M be a manifold, $N \subset M$ a regular submanifold. For $p \in N$, show that $Y \in T_pM$ belongs to T_pN if and only if $Yf = 0$ for every $f \in C^\infty(M)$ vanishing on N.

3. If $X, Y, Z \in VF(M)$, show that $L_X L_Y Z - L_Y L_X Z = L_{[X,Y]} Z$.

4. For $\nu \in \Omega^k(M)$ and $n > 1$, let $\nu^n = \nu \wedge \cdots \wedge \nu$ (n times). For $X \in VF(M)$ and $n > 1$, show that $L_X(\nu^n) = n\nu^{n-1} \wedge L_X \nu$ if k is even, but $L_X(\nu^n) = 0$ if k is odd.

5. Let $X, Y \in VF(\mathbf{R}^3)$ satisfy $X(\mathbf{u}) \cdot \mathbf{u} = 0 = Y(\mathbf{u}) \cdot \mathbf{u}$ for all $\mathbf{u} \in \mathbf{R}^3$. Prove that $Z = [X, Y]$ also satisfies $Z(\mathbf{u}) \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in \mathbf{R}^3$.

6. If M, N are compact manifolds, prove that $\chi(M \times N) = \chi(M)\chi(N)$.

7. Prove that any Lie group G is orientable.

8. If G is a compact Lie group of dimension > 0, prove that $\chi(G) = 0$.