As before, only the starred problems are to be handed in.

1. Let $\Sigma$ be the set of ideals in a ring $R$ consisting entirely of zero-divisors. Show that $\Sigma$ contains maximal elements, which are prime ideals. Conclude that the set of zero-divisors is a union of prime ideals. Give an example where it is not an ideal.

2. Show that $\mathbb{A}^n$ with the Zariski topology is compact. Show the same for any affine variety $V \subset \mathbb{A}^n$ with the Zariski topology, that is, the subspace topology from $\mathbb{A}^n$.

3. Let $I, J \subset K[x_1, \ldots, x_n]$. (a) Show that $V(I) \cap V(J) = V(I + J)$. (b) If $I$ and $J$ are radical, must $I + J$ be radical? Hint: consider plane curves intersecting with multiplicity.

4. (a) Show that an irreducible variety is connected in the Zariski topology. (b) Give an example to show that the converse is false. (c) In fact, define the graph of a variety to be the diagram with one point for each irreducible component, with a line connecting two points if and only if the corresponding irreducible components intersect. Show that a variety is connected if and only if its graph is connected.

5. Show that if $K$ is algebraically closed and $f \in K[x_1, \ldots, x_n]$ for $n \geq 2$, then $V(f)$ is infinite.

6. Show that the affine space $\mathbb{A}^n$ over an infinite field $K$ is irreducible.

7. Let $V \subset \mathbb{A}^m$, $W \subset \mathbb{A}^n$ be varieties.
   (a) Show that $V \times W \subset \mathbb{A}^{m+n}$ is a variety.
   (b) Show that if $V$ and $W$ are irreducible, then so is $V \times W$.
   Hint: recall the proof in topology that a product of connected spaces is connected.

8. Suppose a projective hypersurface of degree $d$ in $\mathbb{P}^5$ intersects the Veronese surface in a smooth curve. What is its genus?

9. Show that there is a linear isomorphism $\mathbb{A}^6 \rightarrow \{\text{symmetric } 3 \times 3 \text{ matrices}\}$ such that the cone on the Veronese surface corresponds to the symmetric matrices of rank $\leq 1$. Hint 1: What equations cut out the Veronese surface (or its cone)? Alternative hint 2: For a finite-dimensional vector space $V$, let $S^2V$ be the space of symmetric bilinear forms on $V^*$. Define a map $V \rightarrow S^2V$ by taking $v$ to the form $\langle f, g \rangle = f(v)g(v)$. What is its matrix form?

10. Let $\mathbb{P}^{m-1}$ have homogeneous coordinates $x_i$, $1 \leq i \leq m$, let $\mathbb{P}^{n-1}$ have homogeneous coordinates $y_j$, $1 \leq j \leq n$, and let $\mathbb{P}^{mn-1}$ have homogeneous coordinates $z_{i,j}$. The Segre embedding is the map $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{mn-1}$ defined by $[x_i] \times [y_j] \mapsto [x_iy_j]$. (a) Show that this is indeed well-defined and injective.
     (b) List a finite set of quadratic polynomials in the $z_{i,j}$ which vanish on the image.
     (c) Let $I$ be the homogeneous ideal they generate and show that the image equals $V(I)$.
     (d) If $\mathbb{A}^{mn}$ is identified with the vector space of $m \times n$ matrices, show that the cone on $V(I)$ consists of the matrices of rank $\leq 1$.
     Hint: Done for $m = n = 2$ by Darren in class. Or you could imitate the argument given in class for the Veronese surface. If all this is too hard, try a simple case like $m - 1, n = 2$.

11. Carefully define irreducibility for a projective variety and show that a projective variety $X$ is irreducible if and only if the cone $C(X)$ over it is. That is, if a cone may be expressed nontrivially as a union of affine varieties, then both of them are cones.