THE MANY LIVES OF
A PLANE CUBIC CURVE

Some good sources:
Kirwan, Complex algebraic curves
Miranda, Algebraic curves and
Riemann surfaces
Hartshorne, Algebraic geometry,
IV § 4
The many lives of a complex plane cubic (a.k.a. an elliptic curve)

1. Quotient of \( \mathbb{C} \) by a lattice \( \Lambda \)

\( \Lambda \subset \mathbb{C} \) lattice, i.e. subgroup \( \cong \mathbb{Z} \times \mathbb{Z} \), spanning \( \mathbb{C} \) over \( \mathbb{R} \)

Mult by \( z \in \mathbb{C} \Rightarrow \) can assume

\( \exists \tau \in \mathbb{C} \setminus \mathbb{R} \) \( \Lambda = \{ a + b \tau \mid a, b \in \mathbb{Z} \} \)

\( \mathbb{C}/\Lambda \cong \) torus \( (\mathbb{R}/\mathbb{Z})^2 \)

Compact Riemann surface

= complex 1-mfd
Weierstrass $\beta$-function:
meromorphic $\beta : \mathbb{C} \rightarrow \mathbb{C}$,
$$\beta(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda} \left( \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$
even, poles at $\lambda \in \Lambda$
$$\beta'(z) = \sum_{\lambda \in \Lambda} \frac{-2}{(z-\lambda)^3}$$
odd, poles at $\lambda \in \Lambda$,
periodic i.e. $\forall \lambda \in \Lambda$, $\beta'(z+\lambda) = \beta'(z)$.
In fact $\beta$ periodic too.
Hence $z \mapsto [\beta(z), \beta'(z), 1]$ defines map
$$\mathbb{C} \rightarrow \mathbb{P}^2
\begin{array}{c}
\downarrow \\
\mathbb{C}/\Lambda
\end{array}$$
Holomorphic near $z = 0$:
$$[\frac{1}{z^2} + \text{holo}, \frac{-2}{z^3} + \text{holo}, 1]$$
$$=[z + \text{holo}, -2 + \text{holo}, z^3]$$
Ex1: (a) \( f: \mathbb{C} \to \mathbb{C} \) periodic + holo
\[ \Rightarrow \text{constant} \]
(b) Find counterexample if \( \Lambda \) replaced by \( \mathbb{Z} \) in def. of periodic
(c) \( \exists \ a, b \in \mathbb{C} \) dep. on \( \tau \) s.t.
\[ (\phi')^2 = 4\phi^3 + a\phi + b \]
Hint: Find \( a \) \( (\phi')^2 - 4\phi^3 - a\phi \) has no poles.

Ex 2: Classify automorphisms, that is, invertible holo maps
\( \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda \).
(a) Reduce to case \( f(0) = 0 \)
(b) Lift to universal cover \( \mathbb{C} \),
get home \( \Lambda \to \Lambda \)
(c) Show that \( f \) a linear map is periodic + holo, apply Ex. 1
(d) Classify lattices + linear maps preserving them
Hence image of $C/\Lambda \to \mathbb{P}^2$ lies on plane cubic

$$y^2 = 4x^3 + ax + b.$$  

In fact a bijection, an embedding.

Different generators for $\Lambda$ give different $\tau$ with same $C/\Lambda$.

Change of "$\mathbb{Z}$-basis" $GL(2, \mathbb{Z})$ acts by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$  

Fundamental domain containing one $\tau$ in each orbit is

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\[ e^{\pi i/3} \quad e^{\pi i/3} \]
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II: DOUBLE COVER OF $\mathbb{P}^1$ BRANCHED AT 4 POINTS

Something completely different:

Given $\{x_1, \ldots, x_{2n}\} \subset \mathbb{P}^1 = \text{Riemann sphere},$

\[ \exists \text{ 2-fold covering space} \]

\[ \Pi : F \rightarrow \mathbb{P}^1 \setminus \{x_1, \ldots, x_{2n}\} \]

$\Pi^{-1}$ (small circle around $x_i$) is count

Complete to compact Riemann surf

$\Pi : E \rightarrow \mathbb{P}^1, \ E = F \cup \{y_1, \ldots, y_{2n}\},$

$\Pi(y_i) = x_i.$

Choose small coord disks $C_i$ around $x_i,$ map other disks $D_i \rightarrow C_i$ by $t \mapsto t^2,$ use $D_i$ plus charts for $F$ as atlas for $E.$
What is the genus of $E$?

Triangulate $\mathbb{P}^1$ so vertices $= x_i$.

Then $2 - 2g(E) = \chi(E) = \# v(E) - \# e(E) + \# f(E)$

$= 2n - 2\# e(\mathbb{P}^1) + 2\# f(\mathbb{P}^1)$

$= 2\chi(\mathbb{P}^1) - 2n$

$= 4 - 2n$

$\Rightarrow g = n - 1$

Such Riemann surfaces called elliptic if $g = 1$ (i.e. 4 points), hyperelliptic if $g \geq 1$ (i.e. $\geq 6$ pts)
Cubic equation?
Write $x$ for $[x, 1] \in \mathbb{P}^1$, 
$\infty$ for $[1, 0] \in \mathbb{P}^1$.

Then $GL(2,\mathbb{C})$ acts on $\mathbb{P}^1$ by

$$
(a \ b) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \quad \text{or}
$$

$$
(a \ b) x = \frac{ax + b}{cx + d}
$$

Ex 3: $\forall x_1, x_2, x_3 \in \mathbb{P}^1$, distinct,
$\exists A \in GL(2,\mathbb{C}) \mid Ax_1 = 0,$
$Ax_2 = 1,$
$Ax_3 = \infty,$

unique up to scalar multiple.

Given elliptic curve branched
over $x_1, x_2, x_3, x_4 \in \mathbb{P}^1$, let $\lambda = Ax_4$.
Now branched over $0, 1, \infty, \lambda$. 
Affine plane cubic $C$ given by
or projective
\[ y^2 = x (x - \lambda_2) (x - \lambda_3) \]
maps to $\mathbb{P}^1$. Indeed
\[ \pi : \mathbb{P}^2 \setminus \{010\} \to \mathbb{P}^1 \]
\[ [x, y, z] \mapsto [x, z] \]
and \((\pi |_C)^{-1}(x) = \begin{cases} 2 \text{ pts} & x \neq 0, 1, \infty, \\ 1 \text{ pt} & x = 0, 1, \infty. \end{cases}\]

Clear if $x \neq \infty$: $y = \pm \sqrt{x(x-1)(x-\lambda)}$.

Choice of $\lambda$ is not unique:
can permute $x_1, x_2, x_3, x_4$.
Generators of $S_4$ act by $\lambda^2$
$\lambda \mapsto 1-\lambda, \lambda \mapsto 1/\lambda, \lambda \mapsto \frac{\lambda}{2\lambda-1}$.

But the quantity
\[ j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2} \]

is invariant under these substitutions.

**Fact:** \( \forall j \in \mathbb{C} \exists ! \) smooth plane cubic \( E \), up to projective equivalence (i.e. change of basis in \( \mathbb{P}^2 \)).

**Ex 4:** Talk about singular (i.e. not smooth) plane cubics.