Mathematics GR6262
Algebraic Geometry

Assignment #9
Due April 5, 2023

Say that a closed subset Y of an affine variety X is *principal* (or a *hypersurface*) if $Y = \mathbf{V}(f)$ for some nonzero $f \in k[X]$. Also say, of course, that a closed subset Y of an arbitrary variety X is *locally principal* if there is an open affine cover $X = \bigcup U_i$ with $Y \cap U_i \subset U_i$ principal.

1. A closed $Y \subset X$ is locally principal if and only if the ideal $I_Y \subset \mathcal{O}_{X,x}$ is principal for each $x \in X$.

2. For closed $Y \subset X$, the ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}$ is locally free if and only if Y is locally principal.

3. A point a with nonzero y-coordinate on the affine plane cubic $\mathbf{V}(y^2 - x^3 + x) \subset \mathbf{A}^2$ is locally principal but not principal.

4. If a is the point on the affine plane cubic C as before, show that $\mathcal{O}(a)$ is locally free of rank 1 as a sheaf over C but is not trivial (i.e. $\mathcal{O}(a) \ncong \mathcal{O}$). Hence $\operatorname{Pic} C \neq 0$ is possible for C affine.

5. If $\phi : S \to T$ is a homomorphism of sheaves, then the kernel presheaf K defined by $K(U) := \ker \phi_U$ is a sheaf.

6. (a) A sheaf homomorphism $\phi : S \to T$ over a base X is injective if and only if for each $x \in X$, $\phi_x : S_x \to T_x$ is injective.

(b) Same with injective replaced by surjective.

7. Show that if $X \in \mathbf{P}^n$ is an irreducible hypersurface of degree d, then $\operatorname{Pic}(\mathbf{P}^n \setminus X) \cong \mathbf{Z}/(d)$.