Mathematics G4263y
Algebraic Geometry

Assignment #8
Due Mar. 29, 2023

1. A divisor $\sum a_i D_i$ on $\mathbf{P}^n$ is principal if and only if $\sum a_i \deg D_i = 0$. (You can do this using the theorems proved in class, but it's more stylish to proceed directly from the definition.)

2. Cartier effective $\neq$ effective Cartier: on $X = \mathbf{V}(x^3 - y^2) \subset \mathbf{A}^2$, the effective divisor D given by 1 times the origin (a) is Cartier but (b) is not of the form $Z(f)$ for any regular $f \in k[X]$.

3. Let $f : \mathbf{P}^M \to \mathbf{P}^N$ be a morphism. If $f^*\mathcal{O}(1) \cong \mathcal{O}(d)$, then (a) $d \geqslant 0$, (b) $d = 0$ iff f is constant, and (c) $d = 1$ iff f is the inclusion of a linear subspace.

4. (a) If $S : \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N$ is the Segre embedding, then $S^*\mathcal{O}(1) \cong \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$.

(b) If $V_d : \mathbf{P}^n \to \mathbf{P}^N$ is the degree d Veronese embedding, then $V_d^*\mathcal{O}(1) \cong \mathcal{O}(d)$.

5. The divisor D defined by $a = b = 0$ in the variety $X \subset \mathbf{A}^4$ defined by $ad - bc = 0$ (the cone on a smooth quadric surface) is not locally principal. That is, Weil does not imply Cartier on X. Hint: if $D \cap U = X \cap \mathbf{V}(f) \cap U$ for some $f \in k[a, b, c, d]$ and some open neighborhood U of $0 \in \mathbf{A}^4$, show first that we may assume f homogeneous and $U = \mathbf{A}^4$.

6. We have $\mathrm{Pic}(\mathbf{P}^m \times \mathbf{P}^n) \cong \mathbf{Z} \times \mathbf{Z}$, generated by $\pi_1^*\mathcal{O}(1)$ and $\pi_2^*\mathcal{O}(1)$.

7. (a) Suppose that X, Y are smooth varieties and that there is a rational map $f : X \dashrightarrow Y$ that restricts to an isomorphism $X \setminus A \cong Y \setminus B$, where both A and B have codimension > 1. Then $\mathrm{Pic}\, X \cong \mathrm{Pic}\, Y$.

(b) Give a counterexample when the codimension equals 1.