Mathematics GR6262
Algebraic Geometry

Assignment #7
Due Mar. 22, 2023

1. Must a module of finite length over a local ring R be a vector space over $R/\mathfrak{m}$? Give a proof or counterexample.

2. State and prove a characterization of $V(\operatorname{Ann} M)$ for M a module over $k[x_1, \ldots, x_n]$ of finite length, where k is algebraically closed as usual.

3. If $\phi_d : \mathbf{P}^n \to \mathbf{P}^N$ is the degree d Veronese embedding, then the degree of $\phi_d(\mathbf{P}^n)$ is d^n.

4. (a) The degree of the Segre embedding of $\mathbf{P}^r \times \mathbf{P}^s$ in $\mathbf{P}^n$ is $\binom{r+s}{r}$.

(b) Generalize to $\mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_m}$.

5. For an r-dimensional variety $X \subset \mathbf{P}^n$ with Hilbert polynomial P_X, define the *arithmetic genus* to be $p_a(X) := (-1)^r(P_X(0) - 1)$. Hartshorne: "This is an important invariant which, as we will see later, is independent of the projective embedding of Y."

(a) First of all, $p_a(\mathbf{P}^n) = 0$.

(b) If Y is a plane curve of degree d, then $p_a(Y) = (d-1)(d-2)/2$.

(c) More generally, if H is a hypersurface in $\mathbf{P}^n$ of degree d, then $p_a(H) = \binom{d-1}{n}$.

(d) If $Y \subset \mathbf{P}^r$, $Z \subset \mathbf{P}^s$ are of dimensions d, e respectively, and $Y \times Z \subset \mathbf{P}^n$ is embedded via the Segre embedding $\mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^n$, then

$$p_a(Y \times Z) = p_a(Y)p_a(Z) + (-1)^e p_a(Y) + (-1)^d p_a(Z).$$

6. If a triangle is inscribed in a smooth conic, then the intersections of its sides with the tangents to the conic at the opposite vertices are collinear.