Mathematics GR6262
Algebraic Geometry

Assignment #6
Due Mar. 8, 2023

1. Every constructible set contains a dense open subset of its closure.

2. In a graded ring, a homogeneous ideal $\mathfrak{p}$ is prime if and only if it satisfies the condition that $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ for homogeneous a, b only.

3. Every irreducible curve (over an algebraically closed field) is birational to a plane curve.

4. (a) If $\mathbf{P}^N = \mathbf{P}k[x_0, \ldots, x_m]_d$ is the projective space parametrizing homogeneous polynomials of degree d in $x_0, \ldots, x_m$, then there exists a dense open $U \subset \mathbf{P}^N$ such that $\mathbf{V}(f) \subset \mathbf{P}^m$ is smooth for every $[f] \in U$.

(b) If instead $\mathbf{P}^N$ parametrizes bihomogeneous polynomials in $x_0, \ldots, x_m$ and $y_0, \ldots, y_n$ of bidegree (d, e), then there again exists a dense open $U \subset \mathbf{P}^N$ such that $\mathbf{V}(f) \subset \mathbf{P}^m \times \mathbf{P}^n$ is smooth for every $[f] \in U$.

5. Over a PID, a finitely generated module has finite length if and only if it contains no free submodule. (You may use the classification of finitely generated modules over a PID, which is exactly parallel to the classification of finitely generated abelian groups.)

6. (a) Over $\mathbf{C}[t]$, let $M = \mathbf{C}^2$, where t acts by multiplication by a fixed 2×2 matrix A. What is the length $\ell(M)$ in terms of the entries of A?

(b) Same thing with $\mathbf{C}$ replaced by $\mathbf{R}$.

7. Over an algebraically closed field, every linear subspace of dimension $> (m-1)(n-1)$ of the space of $m \times n$ matrices contains a matrix of rank 1.