Mathematics G4262y
Algebraic Geometry

Assignment #5
Due Feb. 25, 2005

1. (a) Let $G = GL(2) \times GL(2)$ act in the obvious fashion on $\mathbf{P}^1 \times \mathbf{P}^1$. Show that G also acts on $\mathbf{P}^3$ so that the Segre embedding commutes with the action of any $\gamma \in G$. Hint: use the intrinsic formulation.

(b) Explicitly write down a homomorphism $\rho : GL(2) \times GL(2) \to GL(4)$ so that the action from part (a) is just ρ followed by the standard action of $GL(4)$ on $\mathbf{P}^3$. Hint: choose bases.

***2.** Show that there is an action of $GL(2)$ on $\mathbf{P}^n$ preserving (and acting transitively on) the rational normal curve $\mathrm{Ver}_n(\mathbf{P}^1)$. Hint: use the intrinsic formulation again.

***3.** Show that any morphism from an irreducible projective variety to an affine variety must be a constant.

***4.** Show that any hypersurface of degree d in $\mathbf{P}^n$ is taken by Ver_d to the intersection of $\mathrm{Ver}_d\mathbf{P}^n$ with a hyperplane (i.e. a hypersurface of degree 1).

***5.** Show that the complement of any hypersurface in $\mathbf{P}^n$ is affine.

***6.** Show that the intersection of any two curves in $\mathbf{P}^2$ is nonempty.

7. State and prove a generalization of the previous problem to two subvarieties of $\mathbf{P}^n$.

***8.** Show that $\mathbf{P}^1 \times \mathbf{P}^1 \dashleftarrow\dashrightarrow \mathbf{P}^2$ but $\mathbf{P}^1 \times \mathbf{P}^1 \not\cong \mathbf{P}^2$.

9. If a birational map $\phi : X \dashrightarrow Y$ has inverse $\psi : Y \dashrightarrow X$, show that, if $X \times Y$ is identified with $Y \times X$ in the obvious way, then $\Gamma(\phi) = \Gamma(\psi)$.

***10.** Show that the cone (projective or affine) on a rational projective variety is rational. Use this to show that all quadrics (not just nondegenerate ones) are rational.

***11.** For any rational map $\phi : X \dashrightarrow Y$, show that there is a unique maximal open set $U \subset X$ such that ϕ is represented by a morphism defined on U. (This is called the *domain of regularity.*)

12. Let X and Y be varieties. Suppose that there are points $p \in X$ and $q \in Y$ such that the local rings $\mathcal{O}_{p,X}$ and $\mathcal{O}_{q,Y}$ are isomorphic. Show that there are open neighborhoods of p and q that are isomorphic as varieties.

13. *Projection.* Suppose that $U \subset V$ are finite-dimensional vector spaces over K, and W_1, W_2 are both complementary to U, so that $V = U \oplus W_1 = U \oplus W_2$. Show that the rational maps $\phi_1 : \mathbf{P}V \dashrightarrow \mathbf{P}W_1$ and $\phi_2 : \mathbf{P}V \dashrightarrow \mathbf{P}W_2$ are projectively equivalent in that there exists a linear isomorphism $f : \mathbf{P}W_1 \to \mathbf{P}W_2$ such that $\phi_2 = f \circ \phi_1$. (So all that really matters is the choice of U; we call this *projection from* $\mathbf{P}U$.)

***14.** Recall that a *rational normal curve* in $\mathbf{P}^n$ is a curve projectively equivalent to $\mathrm{Ver}_n(\mathbf{P}^1)$. Show that the image, under projection from any point on C, of a rational normal curve $C \subset \mathbf{P}^n$ is a rational normal curve in $\mathbf{P}^{n-1}$. Hint: use **2** above.