Mathematics GR6262
Algebraic Geometry

Assignment #5
Due Mar. 1, 2023

As usual, varieties are over an algebraically closed field k.

1. Recall that a curve is *rational* if it is birational to $\mathbf{P}^1$. Let Y be a smooth rational curve not isomorphic to $\mathbf{P}^1$.

(a) Show that Y is isomorphic to an open subset of $\mathbf{A}^1$.

(b) Show that Y is affine.

(c) Show that $K[Y]$ is a unique factorization domain.

2. Let Y be the curve $y^2 = x^3 - x$ in $\mathbf{A}^2$, and assume that $\operatorname{char} k \neq 2$.

(a) Show that Y is smooth, and hence that $k[Y]$ is integrally closed.

(b) Let $k[x]$ be the subring of $k(Y)$ generated over k by x. Show that it is a polynomial ring and that $k[Y]$ is its integral closure in $k(Y)$.

(c) Show there is an automorphism $\sigma : k[Y] \to k[Y]$ given by $x \mapsto x$ and $y \mapsto -y$. For any $a \in k[Y]$, define the *norm* to be $N(a) = a\,\sigma(a)$. Show that $N(a) \in k[x]$, $N(1) = 1$, and $N(ab) = N(a)\,N(b)$ for any $a, b \in k[Y]$.

(d) Use the norm to show that the units in $k[Y]$ are precisely the nonzero elements of k; that x, $x \pm 1$, and y are irreducible in $k[Y]$; and that $k[Y]$ is not a unique factorization domain. (If you don't use the norm, then you are probably doing it wrong.)

(e) Conclude that Y is not a rational curve and that $k(Y)$ is not a purely transcendental extension of k.

3. (a) Let Y be a smooth complete curve. Show that every nonconstant rational function f on Y defines a finite dominant morphism $\phi : Y \to \mathbf{P}^1$.

(b) Give a counterexample if Y is not smooth.

4. A projective variety $Y \subset \mathbf{P}^n$ is *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $k[x_0, \ldots, x_n]/I_{C(Y)}$ is integrally closed, that is, if the affine cone $C(Y)$ is normal.

(a) If Y is projectively normal, then it is normal.

(b) Let Y be the twisted quartic in $\mathbf{P}^3$ given parametrically by $[t, u] \mapsto [t^4, t^3u, tu^3, u^4]$. Show that Y is isomorphic to $\mathbf{P}^1$, indeed that the morphism is a closed embedding.

(c) Prove that Y from (b) is normal but not projectively normal. Hence projective normality depends on the embedding. See Hartshorne III, Ex. 5.6 for more examples.

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5. Let X be the blow-up of $\mathbf{P}^n$ at the point $[e_n]$, that is, the closure in $\mathbf{P}^{n-1} \times \mathbf{P}^n$ of the rational map $\mathbf{P}^n \to \mathbf{P}^{n-1}$ given by $[x_0, \dots, x_n] \mapsto [x_0, \dots, x_{n-1}]$.

(a) Prove that the morphism $X \to \mathbf{P}^{n-1} \times \mathbf{P}^n \to \mathbf{P}^n$ given by inclusion followed by projection has an inverse that is a rational map but is not a morphism if $n > 1$.

(b) Show that the valuative criterion becomes false if the curve C is replaced by a smooth variety of higher dimension.

6. Using the valuative criterion (or otherwise), show that a proper affine morphism has finite fibers.