As usual, varieties are over an algebraically closed field \( k \).

1. Recall that a curve is rational if it is birational to \( \mathbb{P}^1 \). Let \( Y \) be a smooth rational curve not isomorphic to \( \mathbb{P}^1 \).
   (a) Show that \( Y \) is isomorphic to an open subset of \( \mathbb{A}^1 \).
   (b) Show that \( Y \) is affine.
   (c) Show that \( K[Y] \) is a unique factorization domain.

2. Let \( Y \) be the curve \( y^2 = x^3 - x \) in \( \mathbb{A}^2 \), and assume that \( \text{char } k \neq 2 \).
   (a) Show that \( Y \) is smooth, and hence that \( k[Y] \) is integrally closed.
   (b) Let \( k[x] \) be the subring of \( k(Y) \) generated over \( k \) by \( x \). Show that it is a polynomial ring and that \( k[Y] \) is its integral closure in \( k(Y) \).
   (c) Show there is an automorphism \( \sigma : k[Y] \to k[Y] \) given by \( x \mapsto x \) and \( y \mapsto -y \). For any \( a \in k[Y] \), define the norm to be \( N(a) = a \sigma(a) \). Show that \( N(a) \in k[x] \), \( N(1) = 1 \), and \( N(ab) = N(a)N(b) \) for any \( a, b \in k[Y] \).
   (d) Use the norm to show that the units in \( k[Y] \) are precisely the nonzero elements of \( k \); that \( x, x \pm 1, \) and \( y \) are irreducible in \( k[Y] \); and that \( k[Y] \) is not a unique factorization domain. (If you don’t use the norm, then you are probably doing it wrong.)
   (e) Conclude that \( Y \) is not a rational curve and that \( k(Y) \) is not a purely transcendental extension of \( k \).

3. (a) Let \( Y \) be a smooth complete curve. Show that every nonconstant rational function \( f \) on \( Y \) defines a finite dominant morphism \( \phi : Y \to \mathbb{P}^1 \).
   (b) Give a counterexample if \( Y \) is not smooth.

4. A projective variety \( Y \subset \mathbb{P}^n \) is projectively normal (with respect to the given embedding) if its homogeneous coordinate ring \( k[x_0, \ldots, x_n]/I_C(Y) \) is integrally closed, that is, if the affine cone \( C(Y) \) is normal.
   (a) If \( Y \) is projectively normal, then it is normal.
   (b) Let \( Y \) be the twisted quartic in \( \mathbb{P}^3 \) given parametrically by \( [t, u] \mapsto [t^4, t^3u, tu^3, u^4] \). Show that \( Y \) is isomorphic to \( \mathbb{P}^1 \), indeed that the morphism is a closed embedding.
   (c) Prove that \( Y \) from (b) is normal but not projectively normal. Hence projective normality depends on the embedding. See Hartshorne III, Ex. 5.6 for more examples.

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5. Let $X$ be the blow-up of $\mathbb{P}^n$ at the point $[e_n]$, that is, the closure in $\mathbb{P}^{n-1} \times \mathbb{P}^n$ of the rational map $\mathbb{P}^n \to \mathbb{P}^{n-1}$ given by $[x_0, \ldots, x_n] \mapsto [x_0, \ldots, x_{n-1}]$.

(a) Prove that the morphism $X \to \mathbb{P}^{n-1} \times \mathbb{P}^n \to \mathbb{P}^n$ given by inclusion followed by projection has an inverse that is a rational map but is not a morphism if $n > 1$.

(b) Show that the valuative criterion becomes false if the curve $C$ is replaced by a smooth variety of higher dimension.

6. Using the valuative criterion (or otherwise), show that a proper affine morphism has finite fibers.