Mathematics GR6262
Algebraic Geometry

Assignment #4
Due Feb. 22, 2023

1. (a) If $X \subset Y$ are affine algebraic sets, then every closed embedding $X \to \mathbf{A}^m$ extends to a closed embedding $Y \to \mathbf{A}^{m+n}$ so that $X = Y \cap (\mathbf{A}^m \times 0)$.

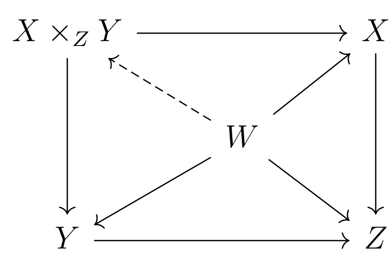
(b) If $\phi : Y \to X$ is a dominant morphism of affine algebraic sets, then for every closed embedding $X \to \mathbf{A}^m$ there exists a closed embedding $Y \to \mathbf{A}^{m+n}$ such that the diagram commutes, where $\pi : \mathbf{A}^{m+n} \to \mathbf{A}^m$ is projection on the first factor:

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{A}^{m+n} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ X & \longrightarrow & \mathbf{A}^m \end{array}$$

2. *The fibered product.* If X, Y, Z are sets and $\phi : X \to Z$, $\psi : Y \to Z$ are maps, then the *fibered product* $X \times_Z Y$ is $\{(x, y) \in X \times Y \mid \phi(x) = \psi(y)\}$.

(a) If X, Y, Z are affine algebraic sets and ϕ, ψ are dominant morphisms, then $X \times_Z Y$ may be given the structure of an affine algebraic set with $k[X \times_Z Y] \cong k[X] \otimes_{k[Z]} k[Y]$.

(b) Show that it satisfies the universal property that, if W is affine and the solid morphisms in the commutative diagram exist, then so does the dotted morphism.



(c) What goes wrong if $X = \mathbf{A}^1 = Y$ and $Z = \mathbf{A}^2$, and instead of being dominant, $\phi(t) = (t, 0)$ and $\psi(t) = (t, t^2)$?

3. *The join or pushout.* If X, Y, Z are sets and $\phi : Z \to X$, $\psi : Z \to Y$ are maps, then the *join* or *pushout* $X \vee_Z Y$ is $(X \sqcup Y)/\sim$, where $\sim$ is generated by $\phi(z) \sim \psi(z)$.

(a) If X, Y, Z are affine algebraic sets and ϕ, ψ are closed embeddings, then $X \vee_Z Y$ may be given the structure of an affine algebraic set. What is $k[X \vee_Z Y]$?

(b) Show that it satisfies a universal property like that for the fibered product, but with all arrows reversed.

(c) What goes wrong if $X = \mathbf{A}^2$, $Y = \mathbf{A}^0$, $Z = \mathbf{A}^1$, and $\phi(z) = (z, 0)$?

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4. Using the definitions and theorems stated in the course so far, show that a finite morphism of algebraic sets is proper.

5. Let X be an algebraic set, $c \in X$. Show that for every $\rho \in T_c X$, $D_\rho(f) = \rho(f(x) - f(c))$ defines a *derivation* $D_\rho : \mathcal{O}_{X,c} \to k$, that is, a k-linear map satisfying the Leibniz rule $D_\rho(fg) = D_\rho(f)g(c) + f(c)D_\rho(g)$. Also show that every such derivation equals D_ρ for some $\rho \in T_c X$.

6. On an algebraic set V, show that $h(x) = \dim T_x V$ is upper semicontinuous.

7. Show that the affine cone on a smooth conic, $\mathbf{V}(uw - v^2) \subset \mathbf{A}^3$, is a normal variety. Hint: let ± 1 act on $k[x, y]$ and its fraction field by $x \mapsto -x$, $y \mapsto -y$. What are the invariant parts?