Mathematics G4262y
Algebraic Geometry

Assignment #3
Due Feb. 11, 2005

1. The subvariety $\mathrm{Ver}_d(\mathbf{P}^1) \subset \mathbf{P}^d$ is called the *rational normal curve*. Choose any $m, n > 0$ such that $m + n = d + 2$. Show that the rational normal curve consists of those points with homogeneous coordinates $[x_0, \ldots, x_n]$ where the $m \times n$ matrix A given by $A_{ij} = x_{i+j-2}$ has rank 1.

***2.** If $\mathbf{P}^3$ has coordinates $[w, x, y, z]$, then the *twisted cubic* $C = \mathrm{Ver}_3(\mathbf{P}^1)$ is the intersection of the three quadrics $V(wy - x^2)$, $V(xz - y^2)$, and $V(wz - xy)$. Show that the intersection of any two of these is the union of C and a line (i.e. a linear subvariety of dimension 1). Hint: work on the affine chart $w = 1$ where the twisted cubic is parametrized by $(1, t, t^2, t^3)$.

***3.** (a) Let $\phi : X \to Y$ be a morphism. Show that for each $P \in X$, ϕ induces a homomorphism $\phi_P^* : \mathcal{O}_{\phi(P),Y} \to \mathcal{O}_{P,X}$ of local rings.

(b) Show that a morphism is an isomorphism if and only if it is a homeomorphism such that ϕ_P^* is an isomorphism for all $P \in X$.

(c) Show that if X is irreducible and $\phi(X)$ is dense in Y, then ϕ_P^* is injective for all $P \in X$.

***4.** Consider the morphism $\phi : \mathbf{P}^1 \to \mathbf{P}^2$ given by $\phi[s, t] = [s^3, st^2, t^3]$ (a cousin of the twisted cubic!). Show that this is a homeomorphism onto its image, but *not* an isomorphism. Hint: an isomorphism would induce an isomorphism on every local ring.

***5.** For another example, suppose the characteristic of K is $p > 0$, and let the *Frobenius morphism* $F : \mathbf{P}^1 \to \mathbf{P}^1$ be given by $F[s, t] = [s^p, t^p]$. Show that this is a morphism and a homeomorphism, but that the inverse is not a morphism.

6. (a) Let $R = K[x_1, \ldots, x_m]$, $S = K[y_1, \ldots, y_n]$, and let $\mathbf{x} \in \mathbf{A}^m$, $\mathbf{y} \in \mathbf{A}^n$. Show that the subalgebra $\{(f, g) \in R \times S \,|\, f(\mathbf{x}) = g(\mathbf{y})\}$ is finitely presented over K. Sketch the corresponding variety.

(b) Same as (a), but for R, S arbitrary finitely presented algebras over K, and $\mathbf{x}$, $\mathbf{y}$ in the corresponding varieties.

(c) If X and Y are affine varieties, $\mathbf{x} \in X$, $\mathbf{y} \in Y$, show that there exists an affine variety Z homeomorphic to the quotient space $X \sqcup Y/\sim$, where the only nontrivial relation in $\sim$ is $\mathbf{x} \sim \mathbf{y}$. That is, Z is X glued to Y at one point.

***7.** Describe a morphism $\phi : \mathbf{P}^2 \to \mathbf{P}^8$ (a cousin of the Veronese embedding) which is injective except that one point has two points in its preimage.

***8.** *The Segre embedding.* Let $\mathrm{Seg}_{m,n} : \mathbf{P}^{m-1} \times \mathbf{P}^{n-1} \to \mathbf{P}^{mn-1}$ be given by $\mathrm{Seg}_{m,n}([x_i], [y_j]) = [x_i y_j]$. (For once, you may find it more convenient to use indices starting with 1 than 0.) By imitating the Veronese arguments, show (a) that this is a morphism, (b) that its image is a projective variety $Z(I)$ where I is an ideal generated by mn quadrics (i.e. homogeneous polynomials of degree 2), and (c) that it is an embedding, that is, an isomorphism onto its image.

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9. Show that the diagonal $\Delta \subset \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ is a projective subvariety of $\mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$, where the latter is regarded as a projective variety in $\mathbf{P}^{n^2-1}$ via the Segre embedding.

10. An example to show that the homogeneous coordinate ring of a variety is not invariant under isomorphism. Let $X = \mathbf{P}^1$, $Y = \mathrm{Ver}_2(\mathbf{P}^1)$. Show that $S(X) \ncong S(Y)$.