Mathematics GR6262
Algebraic Geometry

Assignment #3
Due Feb. 15, 2023

1. Let V be a finite-dimensional vector space, and let $\mathrm{Seg} : \mathbf{P}V \times \mathbf{P}V \to \mathbf{P}(V \otimes V)$ be the Segre embedding. The eigenspaces of the natural involution on $V \otimes V$ decompose it as $V \otimes V = \mathrm{Sym}^2 V \oplus \Lambda^2 V$. Describe, with proof, the intersection of $\mathrm{Seg}(\mathbf{P}V \times \mathbf{P}V)$ with $\mathbf{P}\,\mathrm{Sym}^2 V$ and with $\mathbf{P}\Lambda^2 V$.

2. The image of the Plücker embedding $\mathrm{Pl} : \mathrm{Gr}_2\,\mathbf{A}^4 \to \mathbf{P}^5$ is a smooth quadric hypersurface.

(a) Write down the equation of this hypersurface explicitly.

(b) Describe the image of the locus X_v where the subspace $V \subset \mathrm{Gr}_2\,\mathbf{A}^4$ contains a fixed nonzero $v \in \mathbf{A}^4$, and also the image of the locus X^f where the subspace V is annihilated by a fixed nonzero linear functional $f : \mathbf{A}^4 \to k$.

In the remaining problems, V and W denote varieties over k.

3. If $f : V \to W$ is any morphism, then $\mathrm{id}_V \times f : V \to V \times W$ is a closed embedding. (Call its image the *graph* of f and denote it by $\Gamma(f)$.)

4. (a) If $f : V \to W$ is any morphism with V complete, then $f(V)$ is closed in W and complete.

(b) Give a counterexample when W is a nonseparated prevariety.

5. If an open subset U of a variety V is complete, then $U = V$ or $U = \varnothing$.

6. If $f \in k[V]$ is a regular function, then its graph is closed even as a subset of $V \times \mathbf{P}^1 \supset V \times \mathbf{A}^1$.

7. If V is complete, then $k[V] = k$.

8. If an affine variety is complete, then it is a point.