Mathematics GR6262
Algebraic Geometry

Assignment #2
Due Feb. 8, 2023

1. As presented in class, the Hauptidealsatz states that if X is an affine variety of dimension d over an algebraically closed field k and $0 \neq f \in k[X]$, then every irreducible component of $\mathbf{V}(f) = \{x \in X \mid f(x) = 0\}$ has dimension $d-1$. Use this to prove that if $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, then every irreducible component of $\mathbf{V}(f_1, \ldots, f_m) \subset \mathbf{A}^n$ has dimension $\geq n - m$.

2. Let $X = \mathbf{V}(wz - xy) \subset \mathbf{A}^4$.

(a) Show that X is irreducible of dimension 3.

(b) Show that $k[X]$ is not a UFD.

(c) Show that, inside X, all irreducible components of the closed subset $\mathbf{V}(w) \subset X$ have dimension 2 but are not of the form $\mathbf{V}(g)$ for any $g \in k[X]$.

3. Let $[x_1, \ldots, x_m]$ and $[y_1, \ldots, y_n]$ be homogeneous coordinates on $\mathbf{P}^{m-1}$ and $\mathbf{P}^{n-1}$, respectively. A polynomial $f \in k[x_1, \ldots, x_m, y_1, \ldots, y_n]$ is said to be *bihomogeneous of bidegree* (d, e) if every term is a monomial of degree d in the x_i times a monomial of degree e in the y_j. For example, $3x_1^2 x_2 y_3^7 - 4x_3^3 y_1^2 y_2^5$ is bihomogeneous of bidegree $(3, 7)$.

(a) Show that a bihomogeneous polynomial $f \neq 0$ vanishes on a well-defined hypersurface $\mathbf{V}(f) \subset \mathbf{P}^{m-1} \times \mathbf{P}^{n-1}$. Classify all hypersurfaces of bidegree $(1,1)$ modulo the action of $GL(m) \times GL(n)$. Hint: it may be easier to work in a coordinate-free fashion on $\mathbf{P}U \times \mathbf{P}W$, where U, W are finite-dimensional vector spaces.

(b) Which of the hypersurfaces you found in (a) are singular, and which are smooth? Hint: dehomogenize, then use the rule that $\operatorname{Sing} \mathbf{V}(f) = \mathbf{V}(f) \cap \mathbf{V}(\nabla f)$.

4. (a) Show that $k[\mathbf{A}^n \setminus 0] \cong k[\mathbf{A}^n]$ if $n > 1$.

(b) Show that $\mathbf{A}^n \setminus 0$ is not isomorphic to an affine variety if $n > 1$.

5. (a) Show that $k[\mathbf{P}^{n-1}] \cong k$.
Hint: Use the previous problem or the standard affine cover.

(b) Show that $\mathbf{P}^{n-1}$ is not isomorphic to an affine variety if $n > 1$.

6. If U and V are quasi-projective varieties, show that $\dim(U \times V) = \dim U + \dim V$. Hint: reduce to the affine case, then use Noether normalization.

7. If there is a dominant morphism of quasi-projective varieties $\phi : U \to V$, show that $\dim U \geq \dim V$. (No algebraic space-filling curves.)