1. As presented in class, the Hauptidealsatz states that if $X$ is an affine variety of dimension $d$ over an algebraically closed field $k$ and $0 \neq f \in k[X]$, then every irreducible component of $V(f) = \{ x \in X \mid f(x) = 0 \}$ has dimension $d - 1$. Use this to prove that if $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, then every irreducible component of $V(f_1, \ldots, f_m) \subset \mathbb{A}^n$ has dimension $\geq n - m$.

2. Let $X = V(wz - xy) \subset \mathbb{A}^4$.
   (a) Show that $X$ is irreducible of dimension 3.
   (b) Show that $k[X]$ is not a UFD.
   (c) Show that, inside $X$, all irreducible components of the closed subset $V(w) \subset X$ have dimension 2 but are not of the form $V(g)$ for any $g \in k[X]$.

3. Let $[x_1, \ldots, x_m]$ and $[y_1, \ldots, y_n]$ be homogeneous coordinates on $\mathbb{P}^{m-1}$ and $\mathbb{P}^{n-1}$, respectively. A polynomial $f \in k[x_1, \ldots, x_m, y_1, \ldots, y_n]$ is said to be bihomogeneous of bidegree $(d, e)$ if every term is a monomial of degree $d$ in the $x_i$ times a monomial of degree $e$ in the $y_j$. For example, $3x_1^2x_2y_3^2 - 4x_3^3y_1^2y_2^5$ is bihomogeneous of bidegree $(3, 7)$.
   (a) Show that a bihomogeneous polynomial $f \neq 0$ vanishes on a well-defined hypersurface $V(f) \subset \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$. Classify all hypersurfaces of bidegree $(1, 1)$ modulo the action of $GL(m) \times GL(n)$. Hint: it may be easier to work in a coordinate-free fashion on $PU \times PW$, where $U, W$ are finite-dimensional vector spaces.
   (b) Which of the hypersurfaces you found in (a) are singular, and which are smooth? Hint: dehomogenize, then use the rule that $\text{Sing } V(f) = V(f) \cap V(\nabla f)$.

4. (a) Show that $k[\mathbb{A}^n \setminus 0] \cong k[\mathbb{A}^n]$ if $n > 1$.
   (b) Show that $\mathbb{A}^n \setminus 0$ is not isomorphic to an affine variety if $n > 1$.

5. (a) Show that $k[\mathbb{P}^{n-1}] \cong k$.
   Hint: Use the previous problem or the standard affine cover.
   (b) Show that $\mathbb{P}^{n-1}$ is not isomorphic to an affine variety if $n > 1$.

6. If $U$ and $V$ are quasi-projective varieties, show that $\dim(U \times V) = \dim U + \dim V$. Hint: reduce to the affine case, then use Noether normalization.

7. If there is a dominant morphism of quasi-projective varieties $\phi : U \to V$, show that $\dim U \geq \dim V$. (No algebraic space-filling curves.)