1. Let \( f : \mathbb{P}^M \to \mathbb{P}^N \) be a morphism. If \( f^* \mathcal{O}(1) \cong \mathcal{O}(d) \), show that \( d \geq 0 \), that \( d = 0 \) iff \( f \) is constant, and that \( d = 1 \) iff \( f \) is the inclusion of a linear subspace.

2. If \( S : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N \) is the Segre embedding, show that \( S^* \mathcal{O}(1) \cong \pi_1^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(1) \).

3. Show that the divisor \( D \) defined by \( a = b = 0 \) in the variety \( X \subset \mathbb{A}^4 \) defined by \( ad - bc = 0 \) (the cone on a smooth quadric surface) is not locally principal. That is, Weil does not imply Cartier on \( X \). Hint: if \( D \cap U = X \cap V(f) \cap U \) for some \( f \in k[a,b,c,d] \) and some nonempty open \( U \subset \mathbb{A}^4 \), show that we may assume \( f \) homogeneous. Then seek a contradiction with the previous problem.

4. A linear system \( V \), that is, a finite-dimensional subspace of \( H^0(X,L) \) for some line bundle, is said to be basepoint-free if for all \( x \in X \) there exists \( \sigma \in V \) such that \( \sigma(x) \neq 0 \). In this case, it defines a morphism \( X \to \mathbb{P}^V \) given by \( x \mapsto \text{ev}_x \), or in terms of a basis \( \sigma_i \) by \( x \mapsto [\sigma_i(x)] \). A linear system is complete and is denoted \( |L| \) if \( V = H^0(X,L) \), incomplete otherwise. Show that the complete linear system \( |\mathcal{O}(d)| \) on \( \mathbb{P}^n \) is the degree \( d \) Veronese embedding.

5. If a linear system \( V \) on a projective variety \( X \) defines a closed embedding \( f : X \to \mathbb{P}^N \) but is incomplete, prove that the cone \( C(f(X)) \) is not normal. Hint: recall A14#4.

6. Prove that \( \text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) \cong \mathbb{Z} \times \mathbb{Z} \), generated by \( \pi_1^* \mathcal{O}(1) \) and \( \pi_2^* \mathcal{O}(1) \).

7. (a) Suppose that \( X, Y \) are smooth varieties and that there is a rational map \( f : X \dashrightarrow Y \) with rational inverse \( g : Y \dashrightarrow X \), both regular except on subsets of codimension > 1. Prove that \( \text{Pic} X \cong \text{Pic} Y \).

(b) Give a counterexample when \( f \) is regular except on a subset of codimension 1.