1. Generalize the proof given in class for complete varieties to prove one direction of the *valuative criterion of properness*: If a morphism \( f : X \to Y \) of varieties is proper, then for every smooth curve \( C \), nonempty open \( U \subset C \), and morphism \( U \to X \) such that the composition \( U \to X \to Y \) extends to a morphism \( C \to Y \), then \( U \to X \) extends to \( C \to X \). That is, the dotted arrow exists in the diagram below.

\[
\begin{array}{ccc}
U & \longrightarrow & X \\
\downarrow & & \downarrow \\
C & \longrightarrow & Y
\end{array}
\]

2. Give a counterexample to the above when \( C \) is not smooth.

3. Prove that a subset of a variety is a finite union of locally closed subsets if and only if it is a finite disjoint union of locally closed subsets. (We call both *constructible.*)

4. Let \( X \) be a variety and let \( S \) be the smallest set of subsets of \( X \) that (i) contains all open sets, (ii) is closed under complements, (iii) is closed under finite unions. Prove that \( S \) is the set of constructible sets.

5. Prove that every curve is birational to a plane curve. (Informally, we know from last semester that the latter cannot be smooth if its genus is not of the form \((d-1)(d-2)/2\).)

   Hint: Maybe there is an easier way, but I was helped by KConrad’s answer at [http://mathoverflow.net/questions/21/finite-extension-of-fields-with-no-primitive-element](http://mathoverflow.net/questions/21/finite-extension-of-fields-with-no-primitive-element)

6. Let \( f : X \to Y \) be a morphism of varieties, \( f^*_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \) the induced homomorphism of local rings.

   (a) Show that \( f \) is an isomorphism if and only if it is a homeomorphism and, for all \( x \in X \), \( f^*_x \) is an isomorphism.

   (b) Show that \( f(X) \) is dense in \( Y \) if and only if, for all \( x \in X \), \( f^*_x \) is injective.

7. If \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) is a morphism with \( f^{-1}(0) = \{3, 5\} \) and \( f^{-1}(\infty) = \{2, 4\} \), then what are the possible values of \( f(0) \)? Why? (Assume 2,3,4,5 are all distinct!!!)

8. Let \( C \subset \mathbb{A}^2 \) be the cusp \( x^2 = y^3 \), let \( \pi : X \to \mathbb{A}^2 \) be the blow-up at 0, and let \( \tilde{C} = \pi^{-1}(C \setminus 0) \). Show that \( \tilde{C} \to C \) is the normalization. What about with \( x^2 = y^5 \)?