Mathematics G4263y
Algebraic Geometry
Assignment #14
Due Thursday, January 29, 2015

Most of these exercises are taken from Chapter I of Hartshorne, or from Shafarevich. All
varieties are irreducible over an algebraically closed field \( K \).

1. Recall that a curve is *rational* if it is birational to \( \mathbb{P}^1 \). Let \( Y \) be a smooth rational
curve not isomorphic to \( \mathbb{P}^1 \).
   (a) Show that \( Y \) is isomorphic to an open subset of \( \mathbb{A}^1 \).
   (b) Show that \( Y \) is affine.
   (c) Show that \( K[Y] \) is a unique factorization domain.

2. Let \( Y \) be the curve \( y^2 = x^3 - x \) in \( \mathbb{A}^2 \), and assume that \( \text{char } K \neq 2 \).
   (a) Show that \( Y \) is smooth, and hence that \( K[Y] \) is integrally closed.
   (b) Let \( K[x] \) be the subring of \( K(Y) \) generated over \( K \) by \( x \). Show that it is a
      polynomial ring and that \( K[Y] \) is its integral closure in \( K(Y) \).
   (c) Show there is an automorphism \( \sigma : K[Y] \to K[Y] \) given by \( x \mapsto x \) and \( y \mapsto -y \).
      For any \( a \in K[Y] \), define the norm to be \( N(a) = a \sigma(a) \). Show that \( N(a) \in K[x] \),
      \( N(1) = 1 \), and \( N(ab) = N(a) N(b) \) for any \( a, b \in K[Y] \).
   (d) Use the norm to show that the units in \( K[Y] \) are precisely the nonzero elements of
      \( K \); that \( x \) and \( y \) are irreducible in \( K[Y] \); and that \( K[Y] \) is not a unique factorization
domain. (If you don’t use the norm, you are probably doing it wrong.)
   (e) Conclude that \( Y \) is not a rational curve and that \( K(Y) \) is not a purely transcendental extension of \( K \).

3. Let \( Y \) be a smooth complete curve. Show that every nonconstant rational function \( f \)
on \( Y \) defines a surjective morphism \( \phi : Y \to \mathbb{P}^1 \), and that for every \( P \in \mathbb{P}^1 \), \( \phi^{-1}(P) \) is
finite.

4. A projective variety \( Y \subset \mathbb{P}^n \) is *projectively normal* (with respect to the given embed-
dding) if its homogeneous coordinate ring \( S[Y] \) is integrally closed.
   (a) \( Y \) is projectively normal if and only if the cone \( C(Y) \) is normal.
   (b) If \( Y \) is projectively normal, then it is normal.
   (c) Let \( Y \) be the twisted quartic in \( \mathbb{P}^3 \) given parametrically by \([t, u] \mapsto [t^4, t^3u, tu^3, u^4] \).
      Then \( Y \) is normal but not projectively normal. See Hartshorne III, Ex. 5.6 for more
      examples.
   (d) Show that the curve from \( C \) is isomorphic to \( \mathbb{P}^1 \), indeed that the morphism is an
      isomorphism onto its image. Hence projective normality depends on the embedding.

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5. Let $G$ be a finite group acting on a normal variety $X$ by morphisms. Show that the quotient $X/G$ is normal. (The quotient is defined to be the affine variety with $K[X/G] = K[X]^G$, where the right-hand side is by definition $\{f \in K[X] \mid \forall g \in G, gf = f\}$. It was shown in a student talk last semester that this is finitely generated over $K$; for a refresher, see Harris §10 or Serre, *Algebraic groups and class fields*, III.12.)

6. Show that the surface $X$ defined by $xy = z^2$ in $\mathbb{A}^3$ is normal.

7. Let $X$ be an affine variety and $E$ a finite extension of $K(X)$. Prove that there exists an affine variety $Y$ and a map $f : Y \to X$ with the properties (1) $f$ is proper and surjective; (2) $Y$ is normal; (3) $K(Y) = E$ with $f^* : k(X) \to K(Y) = E$ the inclusion. It is called the normalization of $X$ in $E$.

8. Again let $X$ be the surface $xy = z^2$ in $\mathbb{A}^3$, and let $E = K(X)(\sqrt{x})$. Show that the normalization of $X$ is $E$ is the affine plane, with normalization map of the form $x = u^2$, $y = v^2$, $z = uv$.

9. Sketch a proof of the assertions of Exercise 7 for $X$ an arbitrary curve, not necessarily affine. Prove that $Y$ is complete if $X$ is.