Mathematics GR6262
Algebraic Geometry

Assignment #12

Due May 3, 2023

1. Let C be a smooth projective curve, fix $x_0 \in C$, and let $i : C \to C \times C$ be $i(x) = (x, x)$, $j : C \to C \times C$ be $j(y) = (x_0, y)$, and $\pi_2 : C \times C \to C$ be $\pi_2(x, y) = y$. Also regard $\Delta = i(C)$ and $D = j(C)$ as effective divisors on $C \times C$. Given a vector bundle E over C, use the push-pull and adjunction formulas to give careful, rigorous proofs that, as asserted in class,

(a) $j_* E = (\pi_2^* E) \otimes \mathcal{O}_D$;

(b) $i_*(E \otimes K_C^*) = \pi_2^* E(\Delta) \otimes \mathcal{O}_\Delta$.

2. Use Riemann-Roch for line bundles to prove Riemann-Roch for vector bundles: if E is a vector bundle of rank r and degree d over a smooth complete curve C of genus g, then

$$\chi(C, E) = d + r(1 - g).$$

3. State and prove a formula for the genus of a smooth curve of bidegree (a, b) in $\mathbf{P}^1 \times \mathbf{P}^1$.

4. Use triangulations to give a topological proof, for curves over the complex numbers, of the Riemann-Hurwitz formula: if $\phi : \tilde{C} \to C$ is a morphism of degree $d > 0$ between smooth complete curves of genera $\tilde{g}$ and g, respectively, then

$$2\tilde{g} - 2 = d(2g - 2) + \sum_{y \in \tilde{C}} (o_y \phi - 1).$$

5. Every smooth complete curve of genus 0 is isomorphic to $\mathbf{P}^1$.

6. If a smooth complete curve has a line bundle of degree 1 with 2 independent sections, then it is isomorphic to $\mathbf{P}^1$.

7. Every smooth complete curve of genus 2 is hyperelliptic.