Mathematics GR6262
Algebraic Geometry

Assignment #11
Due April 26, 2023

1. Let $X = \{(x, y) \in \mathbf{A}^2 \mid xy = 0\}$, and let $i, j : \mathbf{A}^1 \to X$ be the inclusions of the x- and y-axes. Show that there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\vec{0}} \longrightarrow \Omega_X \longrightarrow i_*\Omega_{\mathbf{A}^1} \oplus j_*\Omega_{\mathbf{A}^1} \longrightarrow 0.$$

2. If E is a locally free sheaf over a smooth curve, show (without using cohomology or Riemann-Roch) that there exists a line bundle L such that $E \otimes L$ has a nowhere vanishing regular section.

3. Given a vector bundle E of rank r over a smooth complete curve, define $\deg E := \deg \Lambda^r E$. Then in any short exact sequence $0 \to E' \to E \to E'' \to 0$ of locally free sheaves, $\deg E = \deg E' + \deg E''$.

4. (a) Show that on $\mathbf{P}^n$ there is a short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{n+1} \longrightarrow T_{\mathbf{P}^n} \longrightarrow 0,$$

or, dually,

$$0 \longrightarrow T^*_{\mathbf{P}^n} \longrightarrow \mathcal{O}(-1)^{n+1} \longrightarrow \mathcal{O} \longrightarrow 0.$$

(b) Conclude that $K_{\mathbf{P}^n} \cong \mathcal{O}(-n-1)$.

5. Use the adjunction formula to show that a smooth hypersurface $X \in \mathbf{P}^n$ of degree d has:

(a) K_X^{-1} ample if $d < n + 1$ (the "Fano" case);

(b) K_X trivial, that is, $K_X \cong \mathcal{O}_X$, if $d = n + 1$ (the "Calabi-Yau" case); and

(c) K_X ample if $d > n + 1$ (the "general type" case).

6. If $0 \to S' \to S \to S'' \to 0$ is a short exact sequence of sheaves of $\mathcal{O}_X$-modules, and if S' and if S' and S are quasi-coherent, then so is S''.

7. Given a sheaf S over X, let $D(S)$ be its sheaf of discontinuous sections and let $C(S) = D(S)/S$. Recursively define $D^0S = D(S)$, $C^0S = C(S)$, $D^{i+1}S = D(C^iS)$, and $C^{i+1}S = D^{i+1}S/C^iS$. Then $H^{i+j+1}(X, S) = H^i(X, C^jS)$.