Mathematics GR6262
Algebraic Geometry

Assignment #10
Due April 12, 2023

1. Let F be a sheaf over a topological space X.

(a) For $s \in F(U)$, define the *support* to be $\operatorname{Supp} x := \{x \in U \mid s_x \neq 0 \in F_x\}$. Then $\operatorname{Supp} x$ is closed in U.

(b) For F itself, define the *support* to be $\operatorname{Supp} F := \{x \in X \mid F_x \neq 0\}$. Give an example to show that $\operatorname{Supp} F$ need not be closed in X.

2. Let $F = \tilde{M}$ be a quasi-coherent sheaf over an affine algebraic set X.

(a) For $s \in M = H^0(X, F)$, show that $\operatorname{Supp} s = \mathbf{V}(\operatorname{Ann} s)$, where Ann denotes the annihilator.

(b) For M finitely generated over $\mathcal{O}(X)$ (i.e. F coherent), show that $\operatorname{Supp} F = \mathbf{V}(\operatorname{Ann} M)$.

(c) Conclude that the support of a coherent sheaf on a variety is closed.

3. If S, T are coherent sheaves over a variety X, give an example to show that the presheaf $U \mapsto S(U) \otimes_{\mathcal{O}(U)} T(U)$ need not be a sheaf. (The tensor product $S \otimes T$ is defined to be its sheafification.)

4. If $\tilde{M}, \tilde{N}$ are quasi-coherent over an affine X with $\mathcal{O}(X) = R$, then $\tilde{M} \otimes \tilde{N} \cong \widetilde{M \otimes_R N}$.

5. For S, T quasi-coherent over $X \ni x$, the stalks satisfy $(S \otimes T)_x \cong S_x \otimes_{\mathcal{O}_{X,x}} T_x$.

6. If D, E are Cartier divisors on a variety X, then $\mathcal{O}(D + E) \cong \mathcal{O}(D) \otimes \mathcal{O}(E)$.

7. Say that a module M over a local ring R is *invertible* if there exists another module N such that $M \otimes_R N \cong R$. Then a finitely generated M is invertible if and only if it is free of rank 1.

8. A coherent sheaf F over a variety X is locally free of rank 1 if and only if there exists another coherent sheaf G such that $F \otimes G \cong \mathcal{O}$. (We refer to such sheaves as *invertible* and to G as F^{-1}.)