Mathematics GR6262
Algebraic Geometry

Assignment #1
Due Feb. 1, 2023

1. In a topological space X, a subset Z is said to be *locally closed* if every $x \in Z$ has an open neighborhood U_x such that $Z \cap U_x$ is closed in U_x. Show that a subset is locally closed if and only if it is the intersection of an open set and a closed set.

2. Let I be an ideal in $k[x_i] = k[x_1, \ldots, x_n]$, and let I_d be the subspace of I consisting of homogeneous polynomials of degree d. Let the multiplicative group $\mathbf{G}_m$ act on $k[x_i]$ by $\lambda \cdot x_i = \lambda x_i$ for each i. Show that $\mathbf{G}_m \cdot I = I$ if and only if $I = \bigoplus_d I_d$ if and only if I is generated by finitely many homogeneous polynomials. (As always, you may assume that the field k is algebraically closed, though being infinite will suffice here.)

3. (a) With notation as in **2** above, show that for I radical, $\mathbf{G}_m \cdot I = I$ if and only if $\mathbf{G}_m \cdot V(I) = V(I)$.

(b) Give a counterexample if I is not radical.

4. (a) Show that an affine algebraic set $W \subset \mathbf{A}^n$ is reducible if and only if there exist nonzero $f, g \in k[x_i]$ such that $W \subset V(fg)$ but $W \not\subset V(f)$ and $W \not\subset V(g)$.

(b) Show that a projective algebraic set $W \subset \mathbf{P}^n$ is reducible if and only if there exist nonzero homogeneous $f, g \in k[x_i]$ such that $W \subset V(fg)$ but $W \not\subset V(f)$ and $W \not\subset V(g)$.

5. (a) Show that $\mathbf{A}^n$ is irreducible.

(b) Show that $\mathbf{P}^n$ is irreducible.

6. Let $V \subset \mathbf{A}^m$ and $W \subset \mathbf{A}^n$ be affine algebraic sets.

(a) If C is closed in $V \times W$, show that $\{x \in V \mid \{x\} \times W \subset C\}$ is closed in V.

(b) Show that if V and W are irreducible, then so is $V \times W$.

7. (a) If X is a topological space and $X = \bigcup_u U_i$ an open cover, show that C is closed in X if and only if $C \cap U_i$ is closed in U_i for each i.

(b) Let $\pi : \mathbf{A}^{n+1} \setminus \vec{0} \to \mathbf{P}^n$ be $\pi(x_0, \ldots, x_n) = [x_0, \ldots, x_n]$.

For $\mathbf{A}^n_i = \{[x_0, \ldots, x_n] \in \mathbf{P}^n \mid x_i = 1\}$, show that there is an isomorphism making the following diagram commute:

$$\begin{array}{ccc} \mathbf{A}^n \times \mathbf{G}_m & \xrightarrow{\cong} & \pi^{-1}(\mathbf{A}^n_i) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{A}^n_i & \longrightarrow & \mathbf{P}^n. \end{array}$$

(c) If $V \subset \mathbf{P}^n$ is closed and $\pi^{-1}(V)$ is reducible, then V is reducible.

(d) If $V \subset \mathbf{P}^n$ is closed, then it is irreducible if and only if the affine cone $C(V) \subset \mathbf{A}^{n+1}$ is irreducible.