THE MANY LIVES OF A PLANE CUBIC CURVE

Some good sources:

Kirwan, Complex algebraic curves

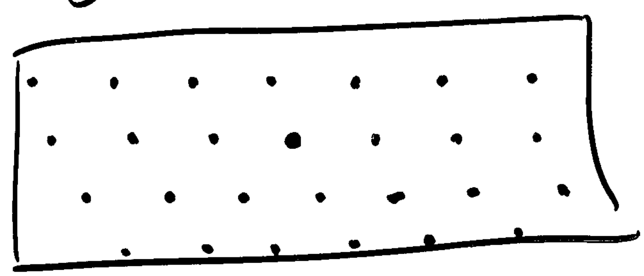
Miranda, Algebraic curves and Riemann surfaces

Hartshorne, Algebraic geometry, $\underline{\text{IV}}$ §4

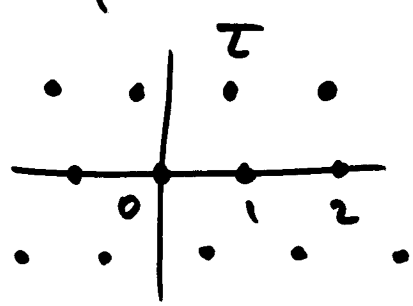
THE MANY LIVES OF A COMPLEX PLANE CUBIC (A.K.A. AN ELLIPTIC CURVE)

I: QUOTIENT OF $\mathbb{C}$ BY A LATTICE Λ

$\Lambda \subset \mathbb{C}$ lattice, i.e. subgroup $\cong \mathbb{Z} \times \mathbb{Z}$, spanning $\mathbb{C}$ over $\mathbb{R}$



Mult by $z \in \mathbb{C} \Longrightarrow$ can assume $\exists\, \tau \in \mathbb{C} \setminus \mathbb{R} \mid \Lambda = \{a + b\tau \mid a, b \in \mathbb{Z}\}$



$\mathbb{C}/\Lambda \cong$  torus $(\mathbb{R}/\mathbb{Z})^2$

Compact $\underbrace{\text{Riemann surface}}_{= \text{ complex 1-mfd}}$

Weierstrass $\wp$-function:

meromorphic $\wp : \mathbb{C} \to \mathbb{C}$,

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$

even, poles at $\lambda \in \Lambda$

$$\wp'(z) = \sum_{\lambda \in \Lambda} \frac{-2}{(z-\lambda)^3}$$

odd, poles at $\lambda \in \Lambda$,

periodic i.e. $\forall \lambda \in \Lambda$, $\wp'(z+\lambda) = \wp'(z)$.

In fact $\wp$ periodic too.

Hence $z \mapsto [\wp(z), \wp'(z), 1]$

defines map

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[dr] & \mathbb{P}^2 \\ & \mathbb{C}/\Lambda \arrow[u] \end{tikzcd}$$

Holomorphic near $z = 0$:

$[\frac{1}{z^2} + \text{holo}, \frac{-2}{z^3} + \text{holo}, 1]$

$= [z + \text{holo}, -2 + \text{holo}, z^3]$

Ex 1: (a) $f : \mathbb{C} \to \mathbb{C}$ periodic + holo $\Rightarrow$ constant

(b) Find counterexample if Λ replaced by $\mathbb{Z}$ in def. of periodic

(c) $\exists\, a, b \in \mathbb{C}$ dep. on τ |

$$(\wp')^2 = 4\wp^3 + a\wp + b$$

Hint: Find a | $(\wp')^2 - 4\wp^3 - a\wp$ has no poles.

Ex 2: Classify automorphisms, that is, invertible holo maps $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$.

(a) Reduce to case $f(0) = 0$

(b) Lift to universal cover $\mathbb{C}$, get homo $\Lambda \to \Lambda$

(c) Show that f - a linear map is periodic + holo, apply Ex. 1

(d) Classify lattices + linear maps preserving them

Hence image of $\mathbb{C}/\Lambda \to \mathbb{P}^2$ lies on plane cubic

$$y^2 = 4x^3 + ax + b.$$

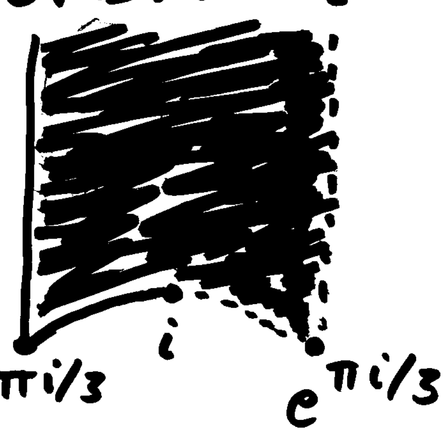
In fact ~~a bijection~~. an embedding

Different generators for Λ give different τ with same $\mathbb{C}/\Lambda$.

Change of "$\mathbb{Z}$-basis" $GL(2,\mathbb{Z})$ acts by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d}$$

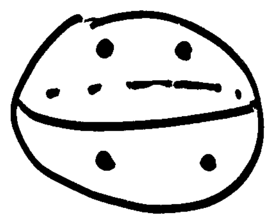
Fundamental domain containing one τ in each orbit is



II : DOUBLE COVER OF $\mathbb{P}^1$ BRANCHED AT 4 POINTS

Something completely different:

Given $\{x_1, \ldots, x_{2n}\} \subset \mathbb{P}^1 =$ Riemann Sphere,



$\exists!$ 2-fold covering space

$\pi : F \to \mathbb{P}^1 \setminus \{x_1, \ldots, x_{2n}\}$ |

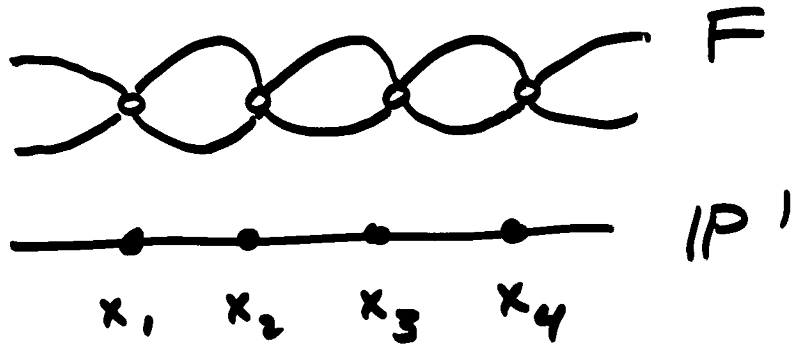
π^{-1}(small circle around x_i) is conn

Complete to compact Riemann surf

$\pi : E \to \mathbb{P}^1$, $E = F \cup \{y_1, \ldots, y_{2n}\}$,

$\pi(y_i) = x_i$.

Choose small coord disks C_i around x_i, map other disks $D_i \to C_i$ by $z \mapsto z^2$, use D_i plus charts for F as atlar for E.



What is the genus of E?

Triangulate $\mathbb{P}^1$ so vertices $= x_i$

Then $2 - 2g(E) = \chi(E)$

$$= \#v(E) - \#e(E) + \#f(E)$$

$$= 2n - 2\#e(\mathbb{P}^1) + 2\#f(\mathbb{P}^1)$$

$$= 2\chi(\mathbb{P}^1) - 2n$$

$$= 4 - 2n$$

$$\Rightarrow \boxed{g = n - 1}$$

Such Riemann surfaces called <u>elliptic</u> if $g = 1$ (i.e. 4 points),
<u>hyperelliptic</u> if $g > 1$ (i.e. ≥ 6 pts)

Cubic equation?

Write x for $[x,1] \in \mathbb{P}^1$,
∞ for $[1,0] \in \mathbb{P}^1$.

Then $GL(2,\mathbb{C})$ acts on $\mathbb{P}^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$ or

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax+b}{cx+d}.$$

Ex 3: $\forall\ x_1, x_2, x_3 \in \mathbb{P}^1$ distinct,
$\exists\ A \in GL(2,\mathbb{C})$ | $Ax_1 = 0$,
$Ax_2 = 1$,
$Ax_3 = \infty$,

unique up to scalar multiple.

Given elliptic curve branched over $x_1, x_2, x_3, x_4 \in \mathbb{P}^1$, Let $\lambda = Ax_4$.
Now branched over $0, 1, \infty, \lambda$.

Affine plane cubic C given by
or projective

$$y^2 z = x(x-z)(x-\lambda z)$$

maps to $\mathbb{P}^1$! Indeed

$$\pi : \mathbb{P}^2 \setminus [0\,1\,0] \longrightarrow \mathbb{P}^1$$
$$[x, y, z] \longmapsto [x, z]$$

and $(\pi|_C)^{-1}(x) = \begin{cases} 2 \text{ pts} & x \neq 0, 1, \infty, \\ 1 \text{ pt} & x = 0, 1, \infty, \end{cases}$

Clear if $x \neq \infty$: $y = \pm\sqrt{x(x-1)(x-\lambda}$

Choice of λ is not unique:
can permute x_1, x_2, x_3, x_4
Generators of S_4 act by
$\lambda \mapsto 1-\lambda$, $\lambda \mapsto 1/\lambda$, $\lambda \mapsto \dfrac{\lambda^2}{2\lambda - 1}$.

But the quantity

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}$$

is invariant under these substitutions.

Fact: $\forall\, j \in \mathbb{C}\ \exists!$ smooth plane cubic E, up to projective equivalence (i.e. change of basis in $\mathbb{P}^2$).

Ex 4: Talk about singular (i.e. not smooth) plane cubics.