Mathematics GU4042
Introduction to Modern Algebra II
Practice Midterm #1
March 2, 2017

Attempt all 7 problems. Each is worth 10 points. Good luck!

1. A theorem from class states that a ring $R$ is a ufd if and only if two conditions (a) and (b) are true. What are they?

2. Prove the converse of Gauss’s Lemma: if $fg = h \in \mathbb{Z}[x]$ is primitive, then $f$ and $g$ are primitive.

3. (a) Proof or counterexample: $\phi : R \to S$ a surjective homomorphism and $R$ a domain imply $S$ a domain.
   
   (b) Proof or counterexample: $\phi : R \to S$ a surjective homomorphism, $R$, $S$ both domains, and $R$ a pid imply $S$ a pid.

4. Which of the following are irreducible in $\mathbb{Z}[x]$ and why? (a) $x^4 + 4x^3 + 6x^2 + 4x + 1$; (b) $x^4 + 4x^3 + 6x^2 + 4x + 2$; (c) $x^4 + x^3 + x^2 + x + 1$; (d) $x^4 + x^3 + x^2 + x$.

5. Prove that if $D$ is an integral domain, then so is $D[x]$ (for $x$ an indeterminate).

6. If $p$ is a prime number and $a \in \mathbb{F}_p$, show that $x^p - a$ is always reducible in $\mathbb{F}_p[x]$.

7. In a ufd $R$, suppose that $a, b, c$ are nonzero nonunits such that $a \mid bc$ and 1 is a gcd of $a$ and $b$. Prove that $a \mid c$. 