Mathematics GU4042
Introduction to Modern Algebra II

Practice Final Examination

May 9, 2017

Attempt all 14 problems. Each is worth 10 points. Good luck!

1. State the primitive element theorem.

2. Which of these properties imply which: being an integral domain, a field, a unique factorization domain, a Euclidean domain, a principal ideal domain, a ring?

3. Prove that if q is an odd prime power, then exactly half the nonzero elements of $\mathbb{F}_q$ are squares.

4. Prove true or false: $\mathbb{Z}[x]/(x^2+3) \cong \mathbb{Z}[x]/(x^2-3)$.

5. (a) How many subfields of $\mathbb{C}$ are isomorphic to $\mathbb{Q}[x]/(x^3+7)$? Prove your answer correct.

(b) How many subfields of $\mathbb{R}$ are isomorphic to $\mathbb{Q}[x]/(x^3+7)$? Prove your answer correct.

6. Factor the following into irreducibles in $\mathbb{Z}[x]$: (a) $x^7+3x^4+18x^2-21$; (b) $2x^4+6x^3+6x^2+2x$; (c) x^6-1. Prove your answers correct.

7. Name an $f \in \mathbb{Q}[x]$ whose splitting field is a Galois extension of $\mathbb{Q}$ of degree 3. Hint: try one of degree 6 first.

8. Describe an extension field of $\mathbb{R}$ of degree 5 or prove that no such field exists.

9. (a) An element a of a ring is *idempotent* if $a^2 = a$. Show that any homomorphism takes idempotents to idempotents.

(b) Describe all homomorphisms $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. Prove your answer correct.

10. (a) What is the factorization of 145 into irreducibles in the Gaussian integers?

(b) Express 145 in all possible ways as a sum of two perfect squares.

11. Describe the group $\mathbb{Z}_{24}^{\times}$. Is it cyclic?

12. (a) Give an example of a non-maximal prime ideal or prove that this is impossible.

(b) Give an example of a non-prime maximal ideal or prove that this is impossible.

13. Let F be an infinite field.

(a) Prove that $f = 0 \in F[x]$ if and only if $f(a) = 0$ for all $a \in F$.

(b) For nonzero $f \in F[x,y]$, prove that $\{a \in F \mid f(a,b) = 0 \text{ for all } b \in F\}$ is a finite set.

14. If E is the splitting field of a polynomial over $\mathbb{Q}$, prove that E contains only finitely many subfields.