Mathematics GU4042
Introduction to Modern Algebra II

Midterm Examination #2

April 13, 2017

READ AND FOLLOW CAREFULLY ALL INSTRUCTIONS BELOW

Turn off all electronic devices.
Brief visits to the men's or women's room are OK, but one at a time only, and cellphones must be left with the instructor.
Write your name, "Modern Algebra, Prof. Thaddeus," and the number of blue books on the cover of each blue book.
Write your name on the attendance sheet when it comes around.
Write all answers and work in your blue books. Do not hand in this sheet.
On each page you use, write the number of the problem *in a circle in the margin.*
You may do more than one problem on a single page. Just put all numbers by their problems.
You may do the problems out of order, but this is discouraged as it can lead to misgrading.
When there is any doubt, state briefly but clearly what statements from the text, lecture, or assignments you are using.
In grading the exams, I will emphasize accuracy, brevity, and clarity. Aim for all three.
Attempt all 7 problems. Each is worth 10 points. Good luck!

1. Let $0 \neq f(x) \in F[x]$, where F is a field. State a necessary and sufficient condition for $f(x)$ to have a double root in its splitting field.

2. Does every automorphism of $\mathbf{R}(y, z)$ extend to an automorphism of $\mathbf{C}(y, z)$? Why or why not? (Here, as usual, $\mathbf{R}(y, z)$ denotes the field of rational functions in two indeterminates y, z with real coefficients, and likewise for $\mathbf{C}(y, z)$.)

3. Let f and g be monic and irreducible in $\mathbf{Q}[x]$. If g has a root in the splitting field E of f over $\mathbf{Q}$, prove that g splits in $E[x]$.

4. Let $\sqrt[6]{3}$ denote the positive real 6th root of 3. Determine, with proof, $\operatorname{Gal} \mathbf{Q}(\sqrt[6]{3})/\mathbf{Q}$. Is the extension Galois? Why or why not?

5. Let $D/E/F$ be field extensions, let $d \in D$, and let $f(x) \in F[x]$ and $g(x) \in E[x]$ be the minimal polynomials of d in F and E, respectively. Prove that $g \mid f$ in $E[x]$.

6. Let E/F be a field extension (not necessarily Galois) with $[E : F]$ prime. Prove that there are no intermediate subfields besides E and F themselves.

7. Let p be a prime number and let $q = p^n$ be some power of p.

(a) Prove that $\operatorname{Aut} \mathbf{F}_p = 1$.

(b) Prove that $\operatorname{Gal} \mathbf{F}_q/\mathbf{F}_p = \operatorname{Aut} \mathbf{F}_q$.

(c) Determine, with proof, $\# \operatorname{Aut} \mathbf{F}_q$, the order of the automorphism group of $\mathbf{F}_q$.