Mathematics GU4042
Introduction to Modern Algebra II

Midterm Examination #1

March 2, 2017

READ AND FOLLOW CAREFULLY ALL INSTRUCTIONS BELOW

Turn off all electronic devices.
Brief visits to the men's or women's room are OK, but one at a time only, and cellphones must be left with the instructor.
Write your name, "Modern Algebra, Prof. Thaddeus," and the number of blue books on the cover of each blue book.
Write your name on the attendance sheet when it comes around.
Write all answers and work in your blue books. Do not hand in this sheet.
On each page you use, write the number of the problem *in a circle in the margin.*
You may do more than one problem on a single page. Just put all numbers by their problems.
You may do the problems out of order, but this is discouraged as it can lead to misgrading.
When there is any doubt, state briefly but clearly what statements from the text, lecture, or assignments you are using.
In grading the exams, I will emphasize accuracy, brevity, and clarity. Aim for all three.
Attempt all 7 problems. Each is worth 10 points. Good luck!

1. Fully and precisely define an *unique factorization domain.*

2. Find all gcd's of 5083 and 2873 in $\mathbf{Z}$. (Here gcd = greatest common divisor.)

3. Let R be a ring, I and J two distinct ideals of R. If R/I and R/J are both fields, prove that $R/(I+J)=0$.

4. An ideal $I \subset R$ is said to be *radical* if for all $r \in R$ and for all $n \geq 0$, $r^n \in I$ implies $r \in I$.

(a) Prove that a prime ideal is radical.

(b) Let $I = (x(x-1)) \subset \mathbf{Q}[x]$. Prove that I is radical.

(c) For I as in (b), prove that I is not prime.

5. For $p \in \mathbf{N}$ prime, prove that $x^{p-1} - 1 = \displaystyle\prod_{i=1}^{p-1}(x-[i]) \in \mathbf{F}_p[x]$.

6. Let D be a unique factorization domain. If two elements x and y of D have gcd 1, must there exist $a, b \in D$ such that $1 = ax + by$? Give a proof or a counterexample with proof.

7. Suppose an integer n is a product of two primes, $n = p_1p_2$, and also a sum of two positive squares, $n = a^2 + b^2$ with $a \neq 0 \neq b$.

(a) Show that $a + bi$ is reducible in $\mathbf{Z}[i]$.

(b) Show that p_1 and p_2 may also be expressed as sums of two squares.