1. Prove that, for any ring $R$, there is an unique homomorphism $\phi : \mathbb{Z} \to R$. If $R$ is an integral domain, what constraints does this place on $\ker \phi$?

2. (a) Prove that if a field $K$ has a subring $D$, then it has a subfield isomorphic to the fraction field $F(D)$. Hint: construct a homomorphism $F(D) \to K$ and show it is well-defined and injective.
   
   (b) Prove that every field $K$ has a subfield isomorphic either to $\mathbb{Q}$ or to $\mathbb{F}_p$ for some prime number $p$.

3. If $f : R \to S$ is a ring homomorphism and $s \in S$ is any element, prove that there is an unique homomorphism $\phi : R[x] \to S$ such that $\phi(x) = s$ and $\phi(r) = f(r)$ for every $r \in R$.

4. Let $p$ be a prime number and $n$ a natural number. Show that $x^n - p!$ is irreducible in $\mathbb{Q}[x]$.

5. As in class, let $\pi_p : \mathbb{Z}[x] \to \mathbb{F}_p[x]$ be $\pi_p(\sum a_i x^i) = \sum [a_i] x^i$, and let the reduction of $f$ modulo $p$ be $\pi_p(f)$.
   
   (a) Prove that if a monic $f \in \mathbb{Z}[x]$ is reducible as a product of polynomials of degrees $k$ and $\ell$, then the same is true of its reduction modulo $p$.
   
   (b) Prove that if $\pi_p(f)$ is irreducible, then so is $f$.

6. Using Eisenstein’s criterion and reduction mod 2 as aids, determine which of the following are irreducible in $\mathbb{Q}[x]$: 
   
   (a) $x^2 + 2345x + 125$; (b) $x^3 + 5x^2 + 10x + 5$; (c) $x^3 + 2x^2 + 3x + 1$; (d) $x^4 + 2x^3 + 2x^2 + 2x + 2$;
   
   (e) $x^4 + 2x^3 + 3x^2 + 2x + 1$; (f) $x^4 + 2x^3 + x^2 + 2x + 1$; (g) $x^5 + x^4 - 4x^3 + 2x^2 + 4x + 1$.

7. Prove that the irreducible elements of the Gaussian integers $\mathbb{Z}[i]$ are $1 + i$, the prime numbers $p \equiv 3 \pmod{4}$, the numbers $a \pm bi$ where $a^2 + b^2 \equiv 1 \pmod{4}$ and $a^2 + b^2$ is prime, and their associates. Hint: given any $c + di \in \mathbb{Z}[i]$, consider the factorization of $c^2 + d^2$.

8. Prove that, if $p \in \mathbb{N}$ is prime and $p \equiv 1 \pmod{4}$, then the expression $p = a^2 + b^2$ guaranteed by Fermat’s theorem is unique up to exchanging $a$ and $b$.

9. Factor the following into irreducibles in $\mathbb{Z}[i]$: (a) $6 - 9i$; (b) $10$; (c) $3 + 4i$.

Many of the above exercises are taken from Artin’s book.