Throughout, $F$ denotes a field. A polynomial $a_0 + \cdots + a_n x^n \in F[x]$ is said to be monic if $a_n = 1$.

1. (a) Prove the ideal correspondence: if $J$ is an ideal in $R$ and $\pi : R \rightarrow R/J$ is the projection, then the map taking $I$ to $I/J$ is a bijection between the set of ideals in $R$ containing $J$ and the set of all ideals in $R/J$. What is the inverse map?

   (b) Prove that, in the above, the prime ideals correspond to the prime ideals, and the maximal ideals to the maximal ideals. (Hint: you should not have to work directly from the definitions.)

2. (a) If $g \in F[x]$, prove that every element of $F[x]/(g)$ is represented by an unique polynomial of degree $< \deg g$ (or by 0, which strictly speaking has no degree).

   (b) If $g \in F_p[x]$, how many elements does $F_p[x]/(g)$ have?

3. Prove that a nonconstant monic polynomial in $F[x]$ is irreducible if and only if it cannot be expressed as a product of two nonconstant monic polynomials. Prove that a monic polynomial in $F[x]$ of degree 2 or 3 is irreducible in $F[x]$ if and only if it has no root. Give a counterexample in degree 4.

4. (a) Prove that $x^2 + x + 1$ is irreducible in $F_p[x]$ if and only if $3 \mid p + 1$.

   Hint: Rule out $3 \mid p$ and $3 \mid p - 1$ using $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

   (b) How many monic polynomials of degree 2 are there in $F_p[x]$? How many of them are irreducible?

   (c) Show that, for any prime $p \in \mathbb{N}$, there exists a field with $p^2$ elements.

5. (a) Show that every maximal ideal in $\mathbb{C}[x]$ equals $(x - r)$ for some $r \in \mathbb{C}$. You may assume the fundamental theorem of algebra, which asserts that every nonconstant polynomial $f(x) \in \mathbb{C}[x]$ has a root, hence (by induction) factors as $f(x) = a(x - r_1) \cdots (x - r_n)$.

   (b) Show that every maximal ideal in $\mathbb{R}[x]$ equals either $(x - r)$ for some $r \in \mathbb{R}$ or $(x^2 + bx + c)$ for some $b, c \in \mathbb{R}$ with $b^2 - 4c < 0$. Hint: factor $f(x)$ in $\mathbb{C}[x]$ and observe that if $f(r) = 0$, then $f(\overline{r}) = 0$ too, and $(x - r)(x - \overline{r}) \in \mathbb{R}[x]$.

   (c) Show that $(x^3 - 2)$ is a maximal ideal in $\mathbb{Q}[x]$.

6. Prove that, for any nonzero ideal $(\beta) \subset \mathbb{Z}[i]$, the quotient $\mathbb{Z}[i]/(\beta)$ is finite. Hint: use the division algorithm to show that every coset is represented by an element of norm $< N(\beta)$.

7. Show that $N : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{N}$ given by $N(a + b\sqrt{2}) = |a^2 - 2b^2|$ satisfies $N(xy) = N(x)N(y)$. Then show that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain by imitating the proof for Gaussian integers.

8. Find a monic greatest common divisor of $x^4 - x^3 - 6x^2 + 3x - 9$ and $x^3 - x^2 - 7x + 3$ in $\mathbb{Q}[x]$. Remember that any associate of a gcd is a gcd, and beware of arithmetic errors!