Mathematics GU4041
Introduction to Modern Algebra

Answers to Practice Midterm #2

November 17, 2016

1. If G is a group, H and N subgroups with $N \lhd G$, then HN is a subgroup, $N \lhd HN$, $H \cap N \lhd H$, and $H/(H \cap N) \cong HN/N$.

2. If $G \not\cong 1$, then there exists $g \in G$ such that $g \neq e$. The subgroup $\langle g \rangle$ generated by g is not 1, so it must be G. We know that every cyclic subgroup (that is, every subgroup generated by a single element) is isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}_n$ for some n (Assignment 7 #2). But $\mathbb{Z}$ has the nontrivial proper subgroup $2\mathbb{Z}$, while $\mathbb{Z}_n$ for $n = ab$ composite has the nontrivial proper subgroup $a\mathbb{Z}_n$. The only remaining possibility is $G \cong \mathbb{Z}_p$ for p prime.

3. Suppose $n \in N_1 \cap N_2$ and $g \in G$. Since $N_1 \lhd G$, $gng^{-1} \in N_1$. Likewise, since $N_2 \lhd G$, $gng^{-1} \in N_2$. Hence $gng^{-1} \in N_1 \cap N_2$. So $N_1 \cap N_2 \lhd G$.

Alternative: Let $\pi_1 : G \to G/N_1$ and $\pi_2 : G \to G/N_2$ be the projection homomorphisms. Then $N_1 \cap N_2 = \ker(\pi_1 \times \pi_2) : G \to G/N_1 \times G/N_2$, and all kernels are normal.

4. By the first isomorphism theorem, $\phi(G) \cong G/\ker\phi$, so $\#G = \#\phi(G)\#\ker\phi$ and hence $\#\phi(G) \mid \#G$. And $\phi(G)$ is a subgroup of H, so by Lagrange's theorem $\#\phi(G) \mid \#H$ also. But $(\#G, \#H) = 1$, so $\#\phi(G) = 1$ and $\phi(G) = \{e\}$.

5. Since x and y are in the same orbit, there exists $g \in G$ such that $y = g \cdot x$. Then $h \in G_y \Leftrightarrow h \cdot y = y \Leftrightarrow h \cdot (g \cdot x) = g \cdot x \Leftrightarrow g^{-1} \cdot (h \cdot (g \cdot x)) = x \Leftrightarrow (g^{-1}hg) \cdot x = x \Leftrightarrow g^{-1}hg \in G_x \Leftrightarrow h \in gG_xg^{-1}$.

6. Let $\phi_1(g) := \phi(g, e)$ and $\phi_2(g) := \phi(e, g)$. Then $\phi_1(gh) = \phi(gh, e) = \phi((g, e)(h, e)) = \phi(g, e)\phi(h, e) = \phi_1(g)\phi_2(g)$, so ϕ_1 is a homomorphism, and similarly for ϕ_2. And $\phi(g_1, g_2) = \phi((g_1, e)(e, g_2)) = \phi(g_1, e)\phi(e, g_2) = \phi_1(g_1)\phi_2(g_2)$, but also $\phi(g_1, g_2) = \phi((e, g_2)(g_1, e)) = \phi(e, g_2)\phi(g_1, e) = \phi_2(g_2)\phi_1(g_1)$.

7. By the orbit-stabilizer theorem, $\#G_g = \#G/\#\mathcal{O}_g$, where $\mathcal{O}_g$ is the conjugacy class of g. We know that the latter consists of all elements of the same cycle type as g, that is, of all transpositions (ij). Since $(ij) = (ji)$, the order of i and j does not matter, so the number of transpositions is the same as the number of unordered pairs in $\langle n \rangle$, namely $n!/2!(n-2)!$. Hence

$$\#G_g = \frac{n!}{\frac{n!}{2!(n-2)!}} = 2(n-2)!.$$