Mathematics GU4041
Introduction to Modern Algebra

Answers to Practice Final

December 20, 2016

1. Any finitely generated abelian group is isomorphic to a direct product of cyclic groups.

2. Yes, since $\#\langle S\rangle \geq \#S = 12$ but $\#\langle S\rangle \,|\, \#G = 20$: the only divisor of 20 that is ≥ 12 is 20 itself, so $\#\langle S\rangle = 20$ and $\langle S\rangle = G$.

3. By Sylow 3, since $210 = 2\cdot3\cdot5\cdot7$, this number is $\equiv 1 \pmod 3$ and divides $70 = 2\cdot5\cdot7$. The only possibilities are 1, 7, 10, and 70.

4. The identity $A \to A$ is a bijection, so $\sim$ is reflexive. If $f : A \to B$ is a bijection, then so is $f^{-1} : B \to A$, so $\sim$ is symmetric. And if $f : A \to B$ and $g : B \to C$ are bijections, then so is $g \circ f : A \to C$, so $\sim$ is transitive.

5. No. We proved that the center $ZG \neq 1$, but $ZG \lhd G$ (A10#1), so G certainly fails to be simple unless $ZG = G$. But then G is abelian, and we proved in class (Thm 3, just before Sylow) that it has a subgroup of order p, which like all subgroups of an abelian group is normal.

6. By Sylow Thm 3 there's 1 Sylow 7-subgroup (hence it's normal) and 1 or 7 Sylow 3-subgroups. All Sylows are of prime order, hence cyclic. If there's 1 Sylow-3, it's normal, so the Main Theorem on Direct Products applies and $G \cong \mathbb{Z}_7 \times \mathbb{Z}_3$. If there are 7, the Main Theorem on Semidirect Products still applies, so $G \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ for some nontrivial action of $\mathbb{Z}_3$ on $\mathbb{Z}_7$ by automorphisms. The generator multiplies by some number whose cube is $1 \pmod 7$, which must be 2 or 4 (since 1 would be the trivial action). But 2 and $4 = 2^2$ lead to isomorphic groups, since the automorphism $\phi(g) = g^2$ of $\mathbb{Z}_3$ exchanges them. Hence there are 2 isomorphism classes, namely $\mathbb{Z}_7 \times \mathbb{Z}_3$ and one nontrivial semidirect product $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

7. $\#\Sigma_4 = 4! = 24 = 2^3 \cdot 3$. A Sylow 2-subgroup is a subgroup of order 8. Note that (1234) acts on $\{1,2,3,4\}$ just as a 90° rotation acts on the vertices $\{e_1, e_2, -e_1, -e_2\}$ of a square, while (24) acts just as reflection in the x-axis does. The Sylow 2-subgroup generated by these elements is therefore isomorphic to D_8. Meanwhile, the Sylow 3-subgroup $\{e, (123), (321)\}$, as a group of prime order, is isomorphic to $\mathbb{Z}_3$.

8. By the Main Theorem on Semidirect Products, there would otherwise be two proper, nontrivial subgroups whose intersection is 1. But every nonidentity element of Q_8 has some power equal to -1, so the intersection of any two proper, nontrivial subgroups contains -1.

9. A composition series for $G \times H$ is $1 \times 1 \lhd G \times 1 \lhd G \times H$, with composition factors G and H. We know from A9#7 that there exists a composition series for $G \times H$ including K. If it's at the beginning or end of the series, it's $\cong 1$ or $G \times H$. If it's in the middle, by Jordan-Hölder, the length is 2, so the series is $1 \lhd K \lhd G \times H$, and by Jordan-Hölder again, $K \cong G$ or H.

10. Since $\sigma\tau = \sigma(\tau\sigma)\sigma^{-1}$, these two elements are conjugate in Σ_n, and we know that conjugate elements factor into disjoint cycles of the same sizes.

11. (a) Let $g \cdot n = gng^{-1}$. Note that $N \lhd G$ implies $gng^{-1} \in N$, so this defines a map $G \times N \to N$. Then $g \cdot (h \cdot n) = g \cdot hnh^{-1} = g(hnh^{-1})g^{-1} = (gh)n(gh)^{-1} = gh \cdot n$ and $e \cdot n = ene^{-1} = n$, so it is an action. And $g \cdot mn = gmng^{-1} = gmg^{-1}gng^{-1} = (g \cdot m)(g \cdot n)$, so the action is by automorphisms.

(b) This makes the action a homomorphism $G \to \text{Aut } \mathbb{Z}_5 \cong \mathbb{Z}_4$. But any homomorphism from a group of odd order to $\mathbb{Z}_4$ is trivial, say by problem 4 on midterm 2. Hence for all $g \in G$, $n \in N$, $gng^1 = n$ and so $gn = ng$. Therefore $N \subset ZG$.

12. The prime factorization of $\#G$ must be $p_1p_2 \cdots p_k$ where the primes p_i are all different. By the classification of finite groups, G is isomorphic to a product of cyclic groups of prime power order, so $G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$. Now claim that, by induction on k, this is isomorphic to $\mathbb{Z}_{p_1p_2\cdots p_k}$. If $k = 1$ this is trivial; if it's true for $k-1$, then $G \cong \mathbb{Z}_{p_1\cdots p_{k-1}} \times \mathbb{Z}_{p_k} \cong \mathbb{Z}_{p_1p_2\cdots p_k}$ by the Chinese Remainder Theorem.

13. Proof 1: We know $[HN : N] \,|\, [G : N]$ and $[H : H \cap N] \,|\, \#H$. Hence $[HN : N]$ and $[H : H \cap N]$ are relatively prime. But they are also equal, since they are the orders of HN/N and $H/(H \cap N)$, which are isomorphic by the Second Isomorphism Theorem. Hence $[H : H \cap N] = 1$, so $H \cap N = H$, so $N < H$.

Proof 2: Consider the projection $\pi : G \to G/N$. Then $\pi|_H : H \to G/N$ is a homomorphism between groups of coprime order, hence trivial by problem 4 on midterm 2. Hence $H \subset \ker \pi = N$.

14. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, and let $S = \{\text{functions } f : G \to \{A, B\}\}$. Then G acts on S by $(g \cdot f)(h) = f(g + h)$, and the problem is to count $\#S/G$. If $g \neq e$, then g has order exactly 3, so $G/\langle g \rangle$ contains 3 cosets, and $f \in S^g$ if and only if f is constant on each coset. So $\#S^g = 2^3$. On the other hand, clearly $S^e = S$, and $\#S = 2^9$. By Burnside's lemma,

$$\#S/G = \frac{1}{\#G} \sum_{g \in G} \#S^g = \frac{1}{9}(2^9 + 8 \cdot 2^3) = \frac{1}{9}(2^9 + 2^6) = 2^6(2^3 + 1)/9 = 64.$$