Mathematics GU4041
Introduction to Modern Algebra

Practice Final Exam

December 20, 2016

1. State the classification of finitely generated abelian groups.

2. If $\#G = 20$ and $S \subset G$ with $\#S = 12$, must $\langle S \rangle = G$? Why or why not?

3. What are the possible numbers of Sylow 3-subgroups in a group of order 210?

4. Let S be a set, PS its power set. For $A, B \in PS$, say $A \sim B$ if there exists a bijection $A \to B$. Prove that $\sim$ is an equivalence relation.

5. Can a group of order p^n, where p is prime and $n > 1$, ever be simple? Why or why not?

6. Classify the groups of order 21 up to isomorphism. How many are there?

7. For each prime p dividing $\#\Sigma_4$, describe the Sylow p-subgroup of Σ_4 in terms of familiar groups.

8. Prove that the quaternion group Q_8 is *not* isomorphic to a semidirect product except in a trivial fashion as $Q_8 \rtimes 1$ or $1 \rtimes Q_8$.

9. If G and H are finite simple groups and $K \lhd G \times H$, prove that K is isomorphic to 1, G, H, or $G \times H$.

10. Prove that if $\sigma, \tau \in \Sigma_n$, then $\sigma\tau$ and $\tau\sigma$ factor into disjoint cycles of the same sizes.

11. (a) If $N \lhd G$, prove that conjugation defines an action of G on N by automorphisms.

(b) If $N \lhd G$, $\#N = 5$, and $\#G$ is odd, prove that $N \subset ZG$, the center of G.

12. Prove that a finite abelian group whose order is not divisible by the square of any prime must be cyclic.

13. If G is a finite group with $H < G$ and $N \lhd G$, and if $[G : N]$ and $\#H$ are relatively prime, prove that $H < N$.

14. A company manufactures 3×3 tiles marked with the letters A and B. They want all periodic patterns (with period 3) to be constructible from their tiles. (See the example below.) However, they do not have to make all 2^9 possibilities: the seams between the tiles are barely visible, so the two configurations shown are equivalent. How many different types of tile do they have to keep in stock?

A	B	B	A	B	B	A	B	B
A	B	A	A	B	A	A	B	A
B	B	B	B	B	B	B	B	B
A	B	B	A	B	B	A	B	B
A	B	A	A	B	A	A	B	A
B	B	B	B	B	B	B	B	B
A	B	B	A	B	B	A	B	B
A	B	A	A	B	A	A	B	A
B	B	B	B	B	B	B	B	B

A	B	B	A	B	B	A	B	B
A	B	A	A	B	A	A	B	A
B	B	B	B	B	B	B	B	B
A	B	B	A	B	B	A	B	B
A	B	A	A	B	A	A	B	A
B	B	B	B	B	B	B	B	B
A	B	B	A	B	B	A	B	B
A	B	A	A	B	A	A	B	A
B	B	B	B	B	B	B	B	B