Mathematics GU4041
Introduction to Modern Algebra

Answers to Midterm Exam #2

November 17, 2016

1. Let G be a group, $K \subset H$ both normal subgroups of G. Then H/K is normal in G/K and $(G/K)/(H/K) \cong G/H$.

2. Since G is simple and $\ker\phi$ is normal, either $\ker\phi = G$ or $\ker\phi = 1$. In the first case $\phi(g) = e$. In the second case ϕ is injective. Because G and H have the same order, it is also surjective. (It is not actually necessary that H be simple.)

3. (a) $\{[0]\} = 70\mathbf{Z}_{70} \lhd 10\mathbf{Z}_{70} \lhd 5\mathbf{Z}_{70} \lhd \mathbf{Z}_{70}$;
(b) $1 \lhd A_6 \lhd \Sigma_6$;
(c) $1 \lhd \{R^{3i} \mid i \in \mathbb{Z}\} \lhd \{R^i \mid i \in \mathbb{Z}\} \lhd D_{12}$;
(d) $1 \lhd \{\pm 1\} \lhd \{\pm 1, \pm i\} \lhd Q_8$.

4. Multiplying both sides by g^{-1} on both the right and the left, we find $g^{-1}ghg^{-1} = g^{-1}hgg^{-1}$ and hence $hg^{-1} = g^{-1}h$. Inverting both sides of $hg^{-1} = g^{-1}h$ and $gh = hg$, we also find $gh^{-1} = h^{-1}g$ and $h^{-1}g^{-1} = g^{-1}h^{-1}$. That is, the inverses of g and h commute with g and h as well as with each other.

For $i \in \mathbb{N}$, we prove $g^i h = hg^i$ by induction on i: the case $i = 0$ is $eh = he$, and if it is true for a given i, then $g^{i+1}h = gg^ih = ghg^i = hgg^i = hg^{i+1}$. For $i, j \in \mathbb{N}$, we similarly prove $g^ih^j = h^jg^i$ by induction on j: the case $j = 0$ is $g^ie = eg^i$, and if it is true for a given j, then $g^ih^{j+1} = g^ih^jh = h^jg^ih = h^jhg^i = h^{j+1}g^i$, using $g^ih = hg^i$.

Since for every $i \in \mathbb{Z}$, either $i \in \mathbb{N}$ or $-i \in \mathbb{N}$, and similarly for j, the general case follows by putting together the results of the last two paragraphs.

5. If ϕ is a homomorphism as stated, then $gh = \phi(1,0)\phi(0,1) = \phi(1,1) = \phi(0,1)\phi(1,0) = hg$. Conversely, if $gh = hg$, define ϕ by $\phi(i,j) = g^ih^j$. Then, using the previous problem, $\phi(i+i', j+j') = g^{i+i'}h^{j+j'} = g^ig^{i'}h^jh^{j'} = g^ih^jg^{i'}h^{j'} = \phi(i,j)\phi(i',j')$, so ϕ is a homomorphism.

6. Suppose first that $n \geq 2p$. Let $g = (1 \cdots p)$ and $h = (p+1 \cdots 2p)$ in Σ_n. Since g, h are disjoint cycles, $gh = hg$. By the previous problem, there is a homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to \Sigma_n$ given by $\phi(i,j) = g^ih^j$. By restricting to $\{1, \ldots, p\}$ and to $\{p+1, \ldots, 2p\}$, we see that $\phi(i,j) = e$ if and only if $i, j \in p\mathbb{Z}$, that is, $\ker\phi = p\mathbb{Z} \times p\mathbb{Z}$. By the first isomorphism theorem, $\operatorname{im}\phi \cong (\mathbb{Z} \times \mathbb{Z})/(p\mathbb{Z} \times p\mathbb{Z}) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

On the other hand, if $n < 2p$, then clearly p^2 does not divide $n!$, so Σ_n cannot have a subgroup of order p^2 by Lagrange's theorem.

7. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be the orbits and let $s_i \in \mathcal{O}_i$. Using the counting formula, we have

$$\#S = \sum_{i=1}^{n} \#\mathcal{O}_i = \sum_{i=1}^{n} \frac{\#G}{\#G_{s_i}} = \#G \sum_{i=1}^{n} \frac{1}{\#G_{s_i}} \leq \#G \sum_{i=1}^{n} 1 = n\,\#G.$$