Mathematics GU4041
Introduction to Modern Algebra

Midterm Examination #2

November 17, 2016

READ AND FOLLOW CAREFULLY ALL INSTRUCTIONS BELOW

Turn off all electronic devices.
Brief visits to the men's or women's room (on the 2nd or 4th floors) are OK, but one at a time only, and leave your phone at the front.
Write your name, "Modern Algebra, Prof. Thaddeus," and the number of blue books on the cover of each blue book.
Write your name on the attendance sheet when it comes around.
Write all answers and work in your blue books. Do not hand in this sheet.
On each page you use, write the number of the problem *in a circle in the margin.*
You may do more than one problem on a single page. Just put all numbers by their problems.
You may do the problems out of order, but this is discouraged as it can lead to misgrading.
When there is any doubt, state briefly but clearly what statements from the text, lecture, or assignments you are using.
In grading the exams, I will emphasize accuracy, brevity, and clarity. Aim for all three.
Attempt all 7 problems. Each is worth 10 points. Good luck!

1. State the third isomorphism theorem.

2. Let G and H be finite simple groups of the same order. Prove that every homomorphism $\phi : G \to H$ except for $\phi(g) = e$ is an isomorphism.

3. Write down composition series for the following groups. You need not give proofs, but make sure that it is crystal clear which subgroups you are referring to.

$$\text{(a) } \mathbf{Z}_{70}; \quad \text{(b) } \Sigma_6; \quad \text{(c) } D_{12}; \quad \text{(d) } Q_8.$$

4. If G is a group, $g, h \in G$, and $gh = hg$, prove that for all $i, j \in \mathbb{Z}$ (not just $\mathbb{N}$), we have $g^i h^j = h^j g^i$.

5. Let G be a group and let $g, h \in G$. Prove that there is a homomorphism $\phi : \mathbf{Z} \times \mathbf{Z} \to G$ satisfying $\phi(1, 0) = g$ and $\phi(0, 1) = h$ if and only if $gh = hg$.

6. For a prime p, prove that Σ_n has a subgroup isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$ if and only if $2p \leq n$.

7. If a finite group G acts on a finite set S with n orbits, prove that $\#S \leq n\,\#G$.