Mathematics GU4041
Introduction to Modern Algebra

Assignment #9

Due Wednesday, November 9, 2016

1. Prove that if $\sigma = (a_1 \cdots a_m) \in \Sigma_n$, then for all $i \in \langle m \rangle$, we have $\sigma^i(a_k) = a_j$ for the unique $j \in \langle m \rangle$ such that $j \equiv k + i \pmod{m}$. Deduce that σ has order $|\sigma| = m$.

2. Prove that the order of $\tau \in \Sigma_n$ equals the least common multiple of the lengths of the cycles in its disjoint cycle decomposition.

3. Find all numbers n such that Σ_7 contains an element of order n.

4. Express (123)(145) and (123)(125) and (23)(12)(23) as products of disjoint cycles. What are their orders?

5. If $\sigma, \tau \in \Sigma_n$, and σ has cycle decomposition

$$\sigma = (a_1 a_2 \cdots a_k)(b_1 b_2 \cdots b_\ell) \cdots (z_1 z_2 \cdots z_m),$$

prove that $\tau\sigma\tau^{-1}$ has cycle decomposition

$$\tau\sigma\tau^{-1} = (\tau(a_1)\tau(a_2) \cdots \tau(a_k))\,(\tau(b_1)\tau(b_2) \cdots \tau(b_\ell)) \cdots (\tau(z_1)\tau(z_2) \cdots \tau(z_m)).$$

Hint: a function is determined by its values — so what do both sides do to each element of $\{1, \ldots, n\}$? Consider first $\tau(a_1)\ldots$

6. Let $\sigma = (12)$ and $\tau = (12345) \in \Sigma_5$. Prove that $\Sigma_5 = \langle \{\sigma, \tau\} \rangle$. Hint: use the previous exercise, plus the fact (proved in class) that Σ_n is generated by the set of all transpositions.

7. If G is a finite group and $H \lhd G$, prove that G has a composition series one of whose terms is H.

8. Let G be a finite group. According to the Jordan-Hölder theorem, if $1 \lhd N \lhd G$ and $1 \lhd H \lhd G$ are both composition series for G, then either $N \cong H$ and $G/N \cong G/H$, or $N \cong G/H$ and $G/N \cong H$. But if only one composition series $1 \lhd N \lhd G$ is given, must there exist a second one $1 \lhd H \lhd G$ such that $N \cong G/H$ and $G/N \cong H$? Give a proof or counterexample (and prove it is a counterexample!).