Mathematics GU4041
Introduction to Modern Algebra

Assignment #8

Due October 31, 2016

Recall that $H < G$ means H is a subgroup of G, $H \lhd G$ means H is a normal subgroup of G, and $1 = \{e\}$ is the trivial group. Dummit & Foote's notation slightly differs.

1. Let $\phi : G \to H$ be any homomorphism. Prove that $J < H$ implies $\phi^{-1}(J) < G$ and $J \lhd H$ implies $\phi^{-1}(J) \lhd G$.

2. Prove that a homomorphism $\phi : G \to H$ has $\ker \phi = 1$ if and only if it is injective.

3. (a) Prove that any index 2 subgroup $H < G$ is normal. (As in Assignment #6, the index $[G : H]$ is the number of left cosets of H in G.)

(b) Does $K \lhd H \lhd G$ imply $K \lhd G$? Give a proof or counterexample (and prove it is a counterexample!).

4. An *automorphism* of a group G is an isomorphism $G \to G$.

(a) Prove that the set Aut G of all automorphisms of G is a group under composition.

(b) Prove that for any $h \in G$, $\phi_h(g) := hgh^{-1}$ is an automorphism. Moreover, $f : G \to \text{Aut } G$ given by $f(h) := \phi_h$ is a homomorphism.

(c) Prove that $f(G) \lhd \text{Aut } G$. (The subgroup Inn $G := f(G)$ is called the *inner automorphism group*, and the quotient Out $G :=$ Aut $G/$Inn G is called the *outer automorphism group*.)

5. Prove that Aut $\mathbf{Z}_n \cong \mathbf{Z}_n^\times$. Hint: any automorphism of $\mathbf{Z}_n$ is determined by the image of [1].

6. Prove that for every group G and every $g \in G$, there is an unique homomorphism $\phi : \mathbf{Z} \to G$ such that $\phi(1) = g$.

7. Prove that if $N \lhd G$ is a normal subgroup of prime index $[G : N] = p$, and if $H < G$, then either (i) $H < N$ or (ii) $G = HN$ and $[H : H \cap N] = p$.

8. Suppose $M \lhd G$ and $N \lhd G$, and also that $G = MN$. Prove that $G/(M \cap N) \cong G/M \times G/N$. Hint: define a homomorphism $\phi : G \to G/M \times G/N$ and apply the first isomorphism theorem.

9. Prove the Fundamental Theorem of Direct Products: Suppose $M \lhd G$ and $N \lhd G$, and also that $G = MN$ and $M \cap N = 1$. Prove that $G \cong M \times N$.