Mathematics GU4041
Introduction to Modern Algebra

Assignment #7

Due October 24, 2016

We say that a group G is *generated* by a subset $S \subset G$ if $G = \langle S \rangle$, where $\langle S \rangle$ denotes the set of all words in elements of S and their inverses.

1. (20 pts) Recall that in class we saw that the dihedral group D_{2n} of rigid motions of a regular n-gon is $\{R^i S^j \mid 0 \leq i \leq n-1,\ 0 \leq j \leq 1\}$, where R is rotation by $2\pi/n$ and S is reflection in the horizontal axis. Hence $D_{2n} = \langle R, S \rangle$. Moreover, we saw that $R^n = e = S^2$ and that $RS = SR^{-1}$.

(a) Prove that for all $i \in \mathbf{Z}$ we have $R^i S = SR^{-i}$.

(b) Prove that every element of the form $R^i S$ has order 2.

(c) Prove that D_{2n} is generated by a set with 2 elements, each of which has order 2.

(d) Suppose n is even and $n > 2$, say $n = 2k$. Prove that R^k commutes with every element of D_{2n}, and that it is the only element besides e that does.

(e) Suppose n is odd and $n > 1$. Prove that there is no element besides e that commutes with every element of D_{2n}.

2. (a) Let G be a group. For any $g \in G$, prove that there is an unique homomorphism $\phi : \mathbf{Z} \to G$ such that $\phi(1) = g$.

(b) Prove that the image of this ϕ is a cyclic group (i.e. a group generated by one element, or equivalently, a group of the form $\{g^n \mid n \in \mathbf{Z}\}$ for some element g).

(c) Prove that any cyclic group is isomorphic either to $\mathbf{Z}$ or to $\mathbf{Z}_n$ for some $n \in \mathbf{N}$.

3. (a) If $\phi : G \to H$ is an isomorphism and $g \in G$, prove that $|g| = |\phi(g)|$.

(b) Prove that D_8 is not isomorphic to Q_8.

4. Let $\phi : G \to H$ be a homomorphism of groups (not necessarily an isomorphism), and let $g \in G$ be an element of order n. What can you say about the order of $\phi(g)$? Why?

5. Let $\phi : G \to H$ and $\psi : G \to H$ be homomorphisms. Is $\{g \in G \mid \phi(g) = \psi(g)\}$ a subgroup? Prove or disprove.

6. Let $C = [-1, 1]^3 \subset \mathbf{R}^3$ be a cube. What do you think is the order of the group G of rigid motions of the cube — that is, linear maps $\phi : \mathbf{R}^3 \to \mathbf{R}^3$ such that $\phi(C) = C$? Explain your reasoning, or how you count. You don't have to be completely rigorous, but be clear and count carefully.