Mathematics GU4041
Introduction to Modern Algebra

Assignment #6

Due October 17, 2016

A subgroup H of a group G is said to have *finite index* if the set G/H of left H-cosets in G is finite. If so, $\#(G/H)$ is called the *index* of H in G and denoted $[G : H]$. (Dummit & Foote use the notation $|G : H|$.) Lagrange's theorem may then be written $\#G = [G : H]\,\#H$, for a finite group G.

1. Suppose $H \subset K \subset G$ are subgroups with $[G : H]$ finite. Prove that $[G : K]$ and $[K : H]$ are also finite and $[G : H] = [G : K]\,[K : H]$. Do not assume G is finite.

2. Suppose H and K are subgroups of finite index in G such that $H \cap K$ is also of finite index. If $[G : H] = m$ and $[G : K] = n$, prove that $\operatorname{lcm}(m, n) \leq [G : H \cap K] \leq mn$. Deduce that, if m and n are relatively prime (that is, have gcd = 1), then $[G : H \cap K] = [G : H]\,[G : K]$.

3. Prove that if H and K are finite subgroups of G whose orders are relatively prime, then $H \cap K = \{e\}$.

4. Let G be a group. Show that the right translations r_h are the *only* maps $G \to G$ commuting with all left translations ℓ_g. That is, if $f : G \to G$ is a map such that, for all $g \in G$, $f \circ \ell_g = \ell_g \circ f$, then there exists $h \in G$ such that $f = r_h$.

5. Let H be a subgroup of G. Define a relation on G by $g \sim g'$ if there exists $h \in H$ such that $g' = gh$. Prove that this is an equivalence relation whose equivalence classes are the left H-cosets.

6. Let H be a subgroup of G. Prove that the map taking x to x^{-1} sends each left H-coset to a right H-coset, and gives a bijection between the left H-cosets and the right H-cosets.

7. (a) Show that if p is prime then $\#\mathbf{Z}_p^\times = p - 1$. Here $\mathbf{Z}_n^\times = \{[a] \in \mathbf{Z}_n \,|\, [a] \text{ has a reciprocal}\}$.

(b) By considering the order of elements of $\mathbf{Z}_p^\times$, prove *Fermat's Little Theorem*: if p is prime, then for all $a \in \mathbf{Z}$, $a^p \equiv a \pmod{p}$.