Mathematics GU4041
Introduction to Modern Algebra

Assignment #5

Due October 10, 2016

1. Let S be a set and G a group. Denote by $F(S,G)$ the set of all functions $f : S \to G$. Define a binary operation on $F(S,G)$ and prove that it is a group. Interpret $F(S,G)$ in the case $S = \mathbf{R}$, $G = \mathbf{R}$.

2. (a) Recall that the *symmetric group* Σ_n is the group of all bijections $f : \langle n \rangle \to \langle n \rangle$, where $\langle n \rangle = \{1, \ldots, n\}$. Prove that $\{f \in \Sigma_n \mid f(1) = 1\}$ is a subgroup of Σ_n.

(b) Let S be any set and let $x \in S$. Recall that the set Bij S of all bijections $S \to S$ is a group (with $\circ$). Prove that $\{f \in \text{Bij } S \mid f(x) = x\}$ is a subgroup of Bij S.

3. It was proved in class that all subgroups of $\mathbf{Z}$ (with $+$) are of the form $k\mathbf{Z}$, where $k \in \mathbf{N}$. State and prove a similar description of all subgroups of $\mathbf{Z} \times \mathbf{Z}_2$. Hint: draw some pictures first.

4. If G is a finite group and $\#G$ is even, prove that G contains an element besides the identity which equals its own inverse. (Hint: consider $\{g \in G \mid g \neq g^{-1}\}$; how many elements does it have?) What is this element for $G = \mathbf{Z}_{2n}$?

5. Let g and h be elements of a group G with identity e. Prove that $gh = hg$ if and only if $h^{-1}gh = g$ if and only if $g^{-1}h^{-1}gh = e$.

6. Prove that if G is a group, and if all $g \in G$ satisfy $g^2 = e$, then G is abelian.

7. Let G and H be groups. Prove that $G \times H$ is abelian if and only G and H are both abelian.