Mathematics GU4041x
Introduction to Modern Algebra

Assignment #3

Due September 26, 2016

In what follows, you may assume anything stated in class except what is being asked: notably, you may use the associative, commutative, and distributive properties of multiplication.

1. Recall that the three Peano axioms, as stated in class, were roughly: (P1) 0 is not a successor; (P2) the successor function is injective; (P3) the axiom of induction. For each $i = 1, 2, 3$, define a set $\mathbf{N}_i$, an element 0_i, and a function $f_i : \mathbf{N}_i \to \mathbf{N}_i$, and prove that it does *not* satisfy the ith Peano axiom but does satisfy the other two. Are such sets unique in the same way that $\mathbf{N}$ is?

2. (Ex 2 from class) Prove that for all $x, y, z \in \mathbf{N}$, $x + z = y + z$ implies $x = y$.

3. (Ex 3 from class) Prove that for all $x \in \mathbf{N}$, $x < x'$.

4. The well-ordering principle says that every nonempty subset of $\mathbf{N}$ contains a least element. Prove that this element is unique.

5. Prove that $\mathbf{N}$ has no greatest element, that is, no element greater than or equal to every element of $\mathbf{N}$.

6. Prove that for all $x, y, z \in \mathbf{N}$, $y \leq z$ implies $xy \leq xz$.

7. For $x, y \in \mathbf{N}$, propose a rule to define x^y similar to those for $x+y$ and $x \cdot y$ given in class. (You don't have to prove that it exists, though it would follow the argument for $x + y$ straightforwardly.)

8. A *sequence* of natural numbers is just a function $a : \mathbf{N} \to \mathbf{N}$. Its values $a(i)$ are frequently denoted a_i. For example, $a_i = i^2 + 1$. For such a sequence, we may define $\sum_{i=0}^{n} a_i$ by the rule

$$\begin{cases} \sum_{i=0}^{0} a_i = a_0, \\ \sum_{i=0}^{n'} a_i = (\sum_{i=0}^{n} a_i) + a_{n'} \end{cases}$$

and then use the axiom of induction to prove that this assigns an unique value to $\sum_{i=0}^{n} a_i$. Assuming this, use the axiom of induction to prove that $2\sum_{i=0}^{n} i = n^2 + n$. (You may assume the standard notation $1 = 0'$, $2 = 1 + 1 = 0''$, $n^2 = n \cdot n$, and so on.)