Mathematics GU4041
Introduction to Modern Algebra

Assignment #2

Due September 19, 2016

The *Axiom of Powers* states the following: if S is any set, there exists a set PS, called the *power set* of S, such that $x \in PS \Leftrightarrow x \subset S$. That is, the elements of PS are precisely the subsets of S. For example, $P\{1,2\} = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.

Given a map $f : S \to T$, define another map $Pf : PS \to PT$ by the rule $Pf(U) := \{f(x) \mid x \in U\}$ for $U \subset S$. That is, $Pf(U) = f(U)$, where $f(U)$ is the image of U under f as defined in class.

1. (a) Is $P\varnothing = \varnothing$? Why or why not?

(b) If $f : S \to T$ is injective, prove that Pf is injective.

(c) Prove that $P\,\mathrm{id}_S = \mathrm{id}_{PS}$.

(d) Prove that $P(f \circ g) = Pf \circ Pg$.

(e) If $f : S \to T$ is bijective (i.e. injective and surjective), prove that Pf is bijective. Hint: use the Main Theorem on Inverses and previous parts.

2. (a) Let $S = \{1, 2, 3\}$. Define two specific functions $f : S \to S$ and $g : S \to S$ and show that $g \circ f \neq f \circ g$.

(b) Same thing with S replaced by the real numbers $\mathbf{R}$ (though we haven't defined them rigorously).

3. Prove that $f \circ (g \circ h) = (f \circ g) \circ h$ whenever f, g, h are functions such that both sides are defined.

4. (a) Prove that $g \circ f$ injective implies f injective.

(b) However, give an example to show that $g \circ f$ injective does not imply g injective.

(c) Prove that $g \circ f$ surjective implies g surjective.

(d) However, give an example to show that $g \circ f$ surjective does not imply f surjective. Hint: could use the same example as in (b).

5. (a) Is the restriction of an injective function necessarily injective? Why or why not?

(b) Is the restriction of an surjective function necessarily surjective? Why or why not?

6. (a) If $f : S \to T$ and $U \subset S$, prove that $U \subset f^{-1}(f(U))$.

(b) Give an example showing that $U \neq f^{-1}(f(U))$ in general.

7. Suppose that $f : S \to T$ and U, V are subsets of S.

(a) Prove that $f(U \cup V) = f(U) \cup f(V)$.

(b) Prove that $f(U \cap V) \subset f(U) \cap f(V)$.

(c) Give an example showing that $f(U \cap V) \neq f(U) \cap f(V)$ in general.