Mathematics GU4041
Introduction to Modern Algebra

Assignment #14

Due December 14, 2016

The following theorem, which was proved in class, should be useful:
Let K act on N and K' act on N' by group automorphisms. If $\psi : N \to N'$ and $\phi : K \to K'$ are isomorphisms satisfying $\psi(k \cdot n) = \phi(k) \cdot \psi(n)$ for all $k \in K$ and $n \in N$, then $\beta : N \rtimes K \to N' \rtimes K'$ defined by $\beta(n, k) := (\psi(n), \phi(k))$ is an isomorphism.

1. Let $\operatorname{Bij} \mathbf{R}^3$ be the group of all bijections $\mathbf{R}^3 \to \mathbf{R}^3$.

(a) Prove that the subset of $\operatorname{Bij} \mathbf{R}^3$ consisting of *translations*, that is, bijections $f(\vec{v}) = \vec{w} + \vec{v}$ for a constant $\vec{w}$, is a subgroup isomorphic to the additive group $\mathbf{R}^3$.
Hint: realize it as the image of a homomorphism.

(b) Prove that the subset $SO(3) \subset \operatorname{Bij} \mathbf{R}^3$ consisting of *rotations*, that is, bijections $f(\vec{v}) = A\vec{v}$ for a 3×3 matrix A such that $A^T A = I$ and $\det A = 1$, is also a subgroup.

(c) Define an action of $SO(3)$ on $\mathbf{R}^3$ by group automorphisms and prove that the subset $\operatorname{Aff} \mathbf{R}^3 \subset \operatorname{Bij} \mathbf{R}^3$ consisting of bijections $f(\vec{v}) = \vec{w} + A\vec{v}$ is a subgroup isomorphic to a semidirect product $\mathbf{R}^3 \rtimes SO(3)$.

2. Let $G = \mathbf{Z}_2 \times \mathbf{Z}_2$. Prove that any permutation of the 3 non-identity elements of G defines an automorphism, so that $\operatorname{Aut} G \cong \Sigma_3$.

3. Prove that $\mathbf{Z}_5^\times \cong \mathbf{Z}_4$. (In fact, we'll prove next semester that $\mathbf{Z}_n^\times$ is cyclic whenever n is a prime power.)

4. Let $p < q$ be two primes. Show that any group of order pq is isomorphic to a semidirect product $\mathbf{Z}_q \rtimes \mathbf{Z}_p$ for some action of $\mathbf{Z}_p$ on $\mathbf{Z}_q$.

5. Classify all of the groups of order 55 up to isomorphism. How many are there? Which are abelian?

6. Classify all of the groups of order 20 up to isomorphism. How many are there? Which are abelian?