Mathematics GU4041  
Introduction to Modern Algebra  
Assignment #13  
Due December 7, 2016

1. List all 12 elements of \( A_4 \). Show that \( A_4 \) is not simple by exhibiting a proper normal subgroup.  
   Hint: look back at the classification of groups of order 12.

2. (a) For \( n \geq 3 \), show that \( \Sigma_n \) has no normal subgroup of order 2.  
   (Careful, it says order 2, not index 2!)  
   (b) For \( n \geq 5 \), show that \( A_n \) and 1 are the only proper normal subgroups of \( \Sigma_n \).  
   Hint: use the Second Isomorphism Theorem.

3. (a) Show that the center of a direct product is the direct product of the centers:  
   \[ Z(G_1 \times \cdots \times G_n) = ZG_1 \times \cdots \times ZG_n. \]
   (b) Show that \( G_1 \times \cdots \times G_n \) is abelian if and only if each \( G_i \) is.

4. (a) For \( A \) an abelian group and \( n \in \mathbb{N} \), show that the set \( A(n) \) of all elements whose order is finite and divides \( n \) is a subgroup.  
   (b) If \( A \cong B \), show that \( A(n) \cong B(n) \).  
   (c) Show that \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \not\cong \mathbb{Z}_4 \times \mathbb{Z}_4 \).

5. Show, using the classification of finite abelian groups, that there are exactly 10 isomorphism classes of abelian groups of order 400. Hint: what are the possible Sylow subgroups?

6. Show, using the classification of finite abelian groups, that any finite abelian group is either cyclic or contains a subgroup isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_p \) for some prime \( p \).